SMEM

PR3 Handbook: Further Explorations

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Table of Contents

This booklet was created in English as a joint effort of all project partners.
Translations into Spanish, German, French, Serbian, Greek are available.

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Introduction

Mathematics is pivotal in STEM subjects and vital for nurturing scientific interest in youth. Our project, SMEM (Significant Mathematics for Early Mathematicians), adopts a multifaceted approach. It aims to innovate maths teaching methods, narrow gender gaps in STEM fields, cultivate diverse skills and promote a positive image of maths. It targets children aged three to eight, educators, and those keen on merging maths with play. Approaching education informally, the project's ethos centres on "They learn as we guide," fostering a cycle of experiential learning—Hands-on, Minds-on, Hearts-on, and Talk-on.

The handbooks of the PR1 and PR2 exhibition products of the SMEM project offer comprehensive information for organising both exhibitions and the intended activities that can make use of these materials. The handbooks encompass objectives, content, dynamics, interconnections, and more; providing a holistic pedagogical proposal stemming from our experience and expertise in creating innovative activities and formats. Throughout the process, we repeatedly encountered core challenges within mathematics education, ranging from content and language intricacies to the interplay between physical or virtual manipulation, concept elaboration, and skill stimulation. This journey also prompted us to explore the potential synergy between hands-on and virtual activities.

With this booklet, named *Further Explorations*, we want to continue the conversation ignited between the project partners and the teachers who found our proposal interesting enough to try it out with their students.

We have categorized the SMEM exhibits into five essential topics aimed at the development of mathematical thinking. Each topic is explored in its own dedicated chapter, allowing for a more thorough examination of the subject and providing educators with the opportunity to enhance knowledge and creativity. Additionally, we suggest supplementary activities and tips that you can utilize to design thematic mini-exhibitions or workshops focused on cultivating this particular aspect of mathematical thinking. The five topics are:

- Equilibrium
- Mirrors and symmetries
- Fitting shapes
- Observation and counting
- Paths

Additionally, we have included a final chapter: An example of a SMEM-based workshop.

In the same way that our exhibits aim to foster a joyful and stimulating discovery experience for students, we are confident that the mathematical and pedagogical reflections contained in this booklet will offer a similar opportunity for our teaching colleagues to engage in individual educational research.

To maintain this parallelism, just as we aim to enrich the evolutionary potential of activities proposed to students who aren't the typical beneficiaries of traditional mathematics

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education—often perceived as exclusively for powerful and well-structured minds—we believe that the reflections of teachers working daily in this crucial educational stage hold extraordinary value and elevate the dignity of our profession.

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Balance and Equilibrium

Balance and equilibrium are fundamental aspects of the development of oneself (standing up, coordinating one's own body) and of the evolution of an intuition of the physical world, not only as geometric shapes but also on how these shapes react in the physical world, especially with gravity.

A significant challenge for children lies in connecting abstract mathematical concepts—like geometric shapes and numerical averages—to tangible occurrences, such as achieving perfect equilibrium in objects. At a deeper level lies the fundamental idea that mathematics can explain the physical world and serve as a language for all sciences. While it's not feasible to explicitly convey this notion to young children at the developmental stage we're addressing (as they need a certain level of scientific maturity to grasp such philosophical reflections), concepts like balance and equilibrium serve as effective entry points for introducing this idea to children.

The fundamental notion related to equilibrium is the barycenter. Our exhibits offer multiple experiments designed to unveil the correlation between mathematical averaging and the physics of balance. Teachers can utilize these experiments, tailoring guidance based on the children's age and familiarity. Older children can propose their hypotheses, allowing teachers to steer and refine their understanding.

Background

The barycenter, sometimes referred to as the centre of mass or centroid, is a geometric point associated with two-dimensional shapes, extending to three-dimensional solids. It can be defined purely in geometric terms but also has a physical interpretation that helps our intuition.

Geometrically, the barycenter denotes the average position of all points within the figure, solely considering its shape.

Physically, the barycenter represents the position where we would find a concentrated point mass equivalent to the mass of the figure we consider. To be more precise, if we apply a force at this point, the body undergoes linear acceleration without rotational force. This definition utilizes the physical concept of mass and indirectly refers to forces such as gravity.

This physical definition is probably the one we are most familiar with. It relates to the idea of balancing in equilibrium. Suppose a flat shape is of a physical form (such as a profile cut in a sheet of wood). Then, we can attempt to balance the object on a finger. There exists a singular point— the barycenter—where balance is achievable. By definition, gravity acts on the figure as if it is applied to the barycenter (although, in reality, gravity applies to all the atoms that make up the shape). If the supporting force of our finger is at the same point, then both forces nullify and maintain the equilibrium of the shape.

Another method involves holding the object vertically at its edge and drawing a line downward from the point of balance, marking the barycenter on this line. Repeating this process with other points creates intersecting lines pinpointing the barycenter, as gravity pulls it downward to put it as low as possible.

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Mathematical figures like triangles, rectangles, or cuboids have geometrically constructible equilibrium points. For instance, in a triangle, the barycenter is at the intersection of the medians (we will see why). This construction, however, is based on the geometrical definition.

With two distinct definitions of the barycenter—geometric and physical—we seek to establish their equivalence through a compelling argument or proof.

With the definition of the barycenter as the average position of all points in the shape, we argue that placing a shape atop its barycenter ensures horizontal equilibrium.

To illustrate, imagine each point in the shape as a small particle with weight, akin to tiny balls. On average, for every such particle behind the equilibrium point, there's one in front, balancing out pairs from right to left. These pairs collectively compensate and establish global equilibrium.

The core concept revolves around the notion of an *average*. The underlying intuition is that an average serves as a representative value for a set of values. This principle does not apply only to numerical data, such as heights, weights, or currency. It also includes positional values, which in the plane require two coordinates.

We can start by delving into the arithmetic mean, often called the average of numbers. For two numbers, denoted as a and b, the average, represented as L, is calculated by adding a and b, and dividing the sum by two. It is expressed as:

$$L = \frac{a+b}{2}$$

This number L possesses specific properties: it is equidistant from both a and b. More intriguingly, sum of the distances (with signs) from L to a and b equals 0. For example:

$$(L - a) + (L - b) = \frac{a+b}{2} - a + \frac{a+b}{2} - b = 0.$$

Imagine an infinite rod with no mass extending along the line of real numbers, where unit masses are fixed at positions a and b. To achieve equilibrium, the fulcrum must be precisely positioned at the average point L. Interestingly, the balance of the rod remains unchanged whether it carries two-unit masses at positions a and b, or a single mass of two units positioned solely at point L.

The same principle extends to three numbers. The average, denoted as $L = (a + b + c)/3$, maintains that the sum of its distances (with sign) to these three numbers equals zero. For example, consider the numbers 2, 5, and 11, where the average is $L = (2 + 5 + 11)/3 = 6$, and the distances are summed to $(6 - 2) + (6 - 5) + (6 - 11) = 0$.

Imagine a rod with marks but no weight of its own. We place weights of one unit each at marks 2, 5, and 11. To balance the rod the fulcrum should align with the mark 6. Interestingly, the fulcrum would experience the same force from these three masses as it would from a single mass of three units positioned at mark 6. This principle holds true no matter how many real numbers are considered.

In the scenario of points in the plane, the concept remains similar. We work with two coordinates for each point. For a set of points in the plane—let's say $A = (x_A, y_A)$, $B = (x_B, y_B)$, $C = (x_C, y_C)$—the average position of these three points is a point with coordinates that are the numeric averages of their respective components: $L = ((x_A + x_B + x_C)/3 , (y_A + y_B + y_C)/3)$.

We can compute the average positions for any finite number of points in the plane. In fact, the virtual exhibit Barycenter does exactly that to find the equilibrium point of a drawn figure: It compiles a list of pixels constituting the shape and calculates the average of their x and y coordinates. However, mathematically, this remains an approximation since shapes are composed

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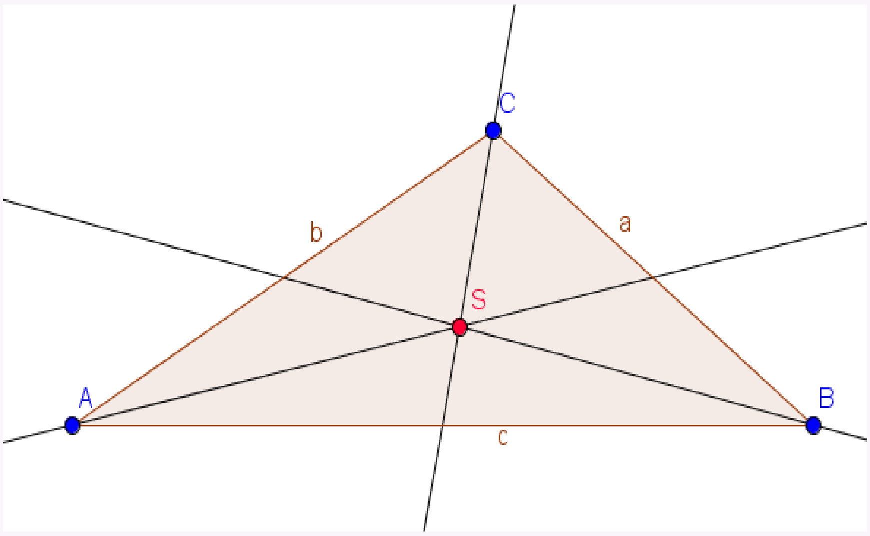
of points, not pixels, which are infinitely small. Calculus offers the infinitesimal rendition of this averaging process, involving an integral as the limit of that sum.

We note that various sets of numbers can yield identical averages. Specifically, computing partial averages allows us to reduce our number list. For example, the average of 2, 5, and 11 equals the average of 2, 8, and 8, which is also equivalent to the weighted average of 2 and 8 with weights ⅓ and ⅔, respectively:

$$\frac{2+5+11}{3} = \frac{2+8+8}{3} = 2 \cdot \frac{1}{3} + 8 \cdot \frac{2}{3} = 6$$

This principle also holds true for points in the plane, coordinate-wise. We can replace two points by a single one at their barycenter (midpoint), given that this point carries the combined mass of the original ones. We refer to this as *the principle of substitution*: We can substitute sections of a figure with their barycenter, to which we assign the weight corresponding to the proportional area we are replacing. An illustration will follow shortly.

An interesting observation arises when considering three points in the plane: the barycenter of three identical punctual masses located at positions A, B, and C coincides with the barycenter of the (complete) triangle formed by these vertices. Note that the latter involves an infinite number of points, while the former entails only three. Furthermore, this barycenter is located at the intersection of the triangle's three medians, which are segments linking a vertex with the midpoint of the opposite side.



Let's illustrate this using the principle of substitution. Considering vertices B and C, we can replace both masses with a single mass of two units positioned at the midpoint of B and C, expressed as $(B + C)/2$. This creates a system comprising a one-unit mass at A and a two-unit mass at the midpoint of B and C. Combining these masses yields a single mass weighting three units and located at the weighted average of the two locations, namely,

$$\frac{1}{3}A + \frac{2}{3} \bullet \frac{B+C}{2} = \frac{A+B+C}{3}$$

This demonstrates that the barycenter lies at ⅔ of the median's length. Due to symmetry, all three medians exhibit this property, indicating that they intersect at the barycenter.

Applying the same substitution principle to the solid triangle, we decompose the surface of triangle ABC into segments parallel to side BC. Each segment resembles a rod with uniform linear density. Substituting each rod with a mass positioned at its midpoint (a segment's barycenter coincides with its midpoint) condenses all segments into points aligned along the median passing through A.

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Consequently, the global barycenter combines all these aligned points along the median, confirming the barycenter's presence somewhere along that median. Due to symmetry, it also resides along the other two medians, thus coinciding with the intersection point of all three medians, confirming it as the same point we identified earlier.

We can leverage this property of triangles to establish methods for calculating the barycenter. Consider a polygonal shape—a figure bounded by straight segments. It is possible to decompose this polygon into triangles, a process known as triangulation. Once we have a triangulation, we compute the barycenter and area for each triangle. Then, we calculate the barycenter of the polygonal shape using substitution. We replace each triangle with its barycenter, assigning weights based on their respective areas. Finally, we compute the (finite!) weighted average of the positions of the triangle barycenters, considering their areas as the weights.

Exhibits related to the barycenter

Within the SMEM project, four exhibits explore the concept of the barycenter. You could combine them to create a thematic session on the topic, with the flexibility to utilize them concurrently or sequentially, which allows children to explore and grasp the relationships between each exhibit.

The Seesaw

The seesaw demonstrates a clear relationship between its two arms: as one rises, the other falls. Initially empty, the centre of gravity rests above the wooden rod. Upon loading, the centre of gravity shifts toward the loaded side, causing an imbalance and tilting.

Usually, the initial assumption is that equilibrium is achieved with equal weight on both sides.

Children are encouraged to balance the seesaw, sparking discussions about the notion of equilibrium — that delicate moment when both arms are poised midway, neither fully raised nor lowered. This unstable state prompts educators to guide children in exploring the factors that make this configuration unique.

At first, children might assume that equilibrium is reached when equal weights are on both sides. However, through experimentation, they learn the importance of the weight's distance from the fulcrum. They discover that to achieve balance, the heavier weight must be placed closer to the middle, while the lighter weight should be farther away. Simple principles, such as 'placing a weight at double the distance compensates for half the mass,' can be discovered by young children. Older children may even uncover the lever law.

Adding more than two weights to the seesaw can deepen understanding. By marking the seesaw with negative and positive values, along with zero at the fulcrum, children can discover that equilibrium happens when the sum of weighted distances (i.e., the sum of the products of the weights by the signed distance) equals zero. This realization comes with reflection and time.

Follow-up activities

The first activity naturally arises as children promptly place bricks on the lever arm to achieve balance. Limiting the number of bricks used (e.g., three or four) encourages an intuitive understanding of the underlying principles.

For the second activity, children place bricks of different colours on opposite sides of the seesaw to achieve balance. It's crucial to position bricks of various sizes but equal weight on identical

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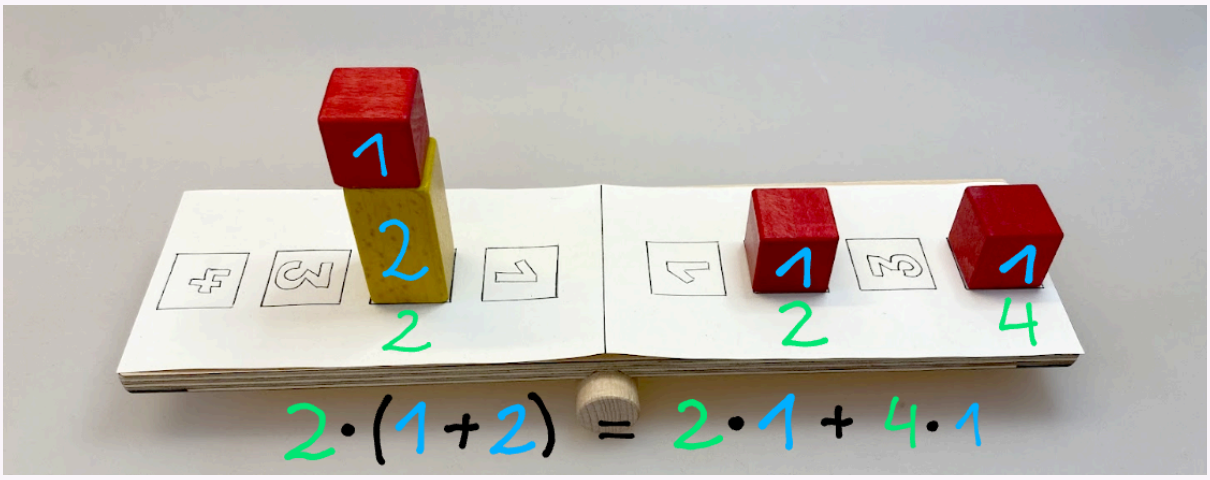
numerical markings (e.g., both sides on mark 2). Through this experiment, children discover that larger-sized bricks are heavier than smaller ones.

The third challenge entails placing cuboids on different numerical markings along the lever arms to achieve balance. If you assign the smallest brick a value of one unit and the larger bricks values of two and four units respectively, equilibrium can be achieved by following just one rule. Namely, the product of the number of units and the number mark where the bricks are on the seesaw must be equal on both sides of the lever arm. For example, placing the four-unit brick in a place marked with number one and the two-unit brick in a place marked with two on the opposite side results in equilibrium.



Looking for an Equilibrium

In the *Looking for an Equilibrium* exhibit, children are tasked with balancing shapes on a wall's edge. After some experimentation, they might effortlessly achieve balance. Some shapes exhibit central symmetry—each point in the shape aligns diametrically with another about a fixed centre, equivalent to a 180-degree rotation. In such cases, the barycenter aligns with the centre of symmetry. When positioned over the wall, it divides the shape into two equal halves (same area, same shape, rotated 180 degrees), thus balancing the shape. However, this is not a general situation, and children should be encouraged to play with non-symmetrical shapes.

Educators can then prompt discussions about what makes this position unique and whether it is somehow special. Initially, children might assume that an equal area on both sides of the wall is necessary for equilibrium, which is incorrect. In the *Seesaw* exhibit, it is noteworthy that both arms didn't require equal weight; instead, adjustments in the distance to the fulcrum were influential. We achieve equilibrium when both arms possess identical leverage, determined by the product of the weight and the distance to the fulcrum.

Here, the setup resembles a lever, with one part of the shape on either side of the wall (fulcrum). The weight in each region corresponds to its area, but what determines the arm lengths? The arm's length is determined by the distance between the region's barycenter (which can be found using tools like virtual exhibits *Creating Umbrellas* or the *Barycenter*) and the contact segment. We can achieve equilibrium when both regions hold the same leverage (area multiplied by the arm length), precisely when the global barycenter rests on the contact segment atop the wall.

In summary, the shape achieves equilibrium *if and only if* the segment touching the wall contains the shape's barycenter. An experiment can illustrate this: Begin by placing the shape in equilibrium on the wall, then introduce a small coin between the shape and the wall at one endpoint of the

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contact segment. The shape maintains balance, now resting on two points: the coin and the opposite endpoint of the segment. By gradually moving the coin towards the opposite endpoint (with the help of a ruler or other flat tool), the shape will balance precisely atop the coin when it reaches the shape's barycenter. You could test this property using a transparent shape from the *Creating Umbrellas* exhibit, with a marked barycenter previously identified by the application.

Follow-up activities

Following up on the exhibit, various engaging classroom activities are available. The children could hunt around the classroom for objects they could balance or do a scavenger hunt for things with similar shapes in their surroundings.

Another challenge involves replicating these objects on paper, cutting out the shapes, and then positioning them on plastic figures, prompting a quest to re-establish equilibrium. Marking a line on the paper where the object balances aids in understanding the equilibrium point. Further exploration can involve examining the figures for symmetry.

A final exercise introduces balancing a rod (such as a broom) with two hands. You could easily find the centre of gravity if you lay a rod on two fingers, one of each hand, and gradually bring both hands together. Surprisingly, the rod keeps in balance. By joining fingers, you pinpoint the rod's centre of gravity.

Creating Umbrellas & Barycenter

The apps Creating umbrellas and Barycenter are closely related. The first one is designed for independent use in an exhibition, featuring a simpler interface tailored for younger children. Conversely, Barycenter extends the functionality of the former, offering the same features and more complex options. This advanced version is better suited for older children and guided use with an educator.

Creating Umbrellas asks to balance a leaf horizontally atop a stick, forming an umbrella. Through the app on a tablet, children can easily draw simple figures, particularly leaves, and visualise their centre of gravity instantly. Positioning the leaf with this point atop the stick ensures equilibrium. Transparent leaf shapes can aid in drawing, and an empirical approach—balancing leaf shapes manually to discover the barycenter—can be compared to the computed barycenter from the app.

Barycenter enables users to draw various shapes on a tablet, exploring the shared barycenter of multiple shapes alongside individual barycenters. Children can choose pre-configured shapes or unleash their creativity by drawing their unique shapes.

Follow-up activities

Children can engage in a hands-on activity by drawing a small shape on the tablet and replicating both the shape and its barycenter onto a piece of paper. Once drawn, the shape is locked on the tablet, so touching the screen will not erase the shape until you click the pencil button. By setting the tablet to maximum brightness, tracing the profile onto a piece of paper placed atop the tablet becomes feasible.

Encourage children to redraw the same shape slightly larger, maintaining a constant distance between their finger/stylus and the shape's edges. Comparing the barycenters of the two shapes should reveal they are identical.

Further exploration can involve suspending various objects on a stick or finger. Examples include balancing a plate on a wooden stick, reminiscent of circus tricks, or spinning a basketball on a finger.

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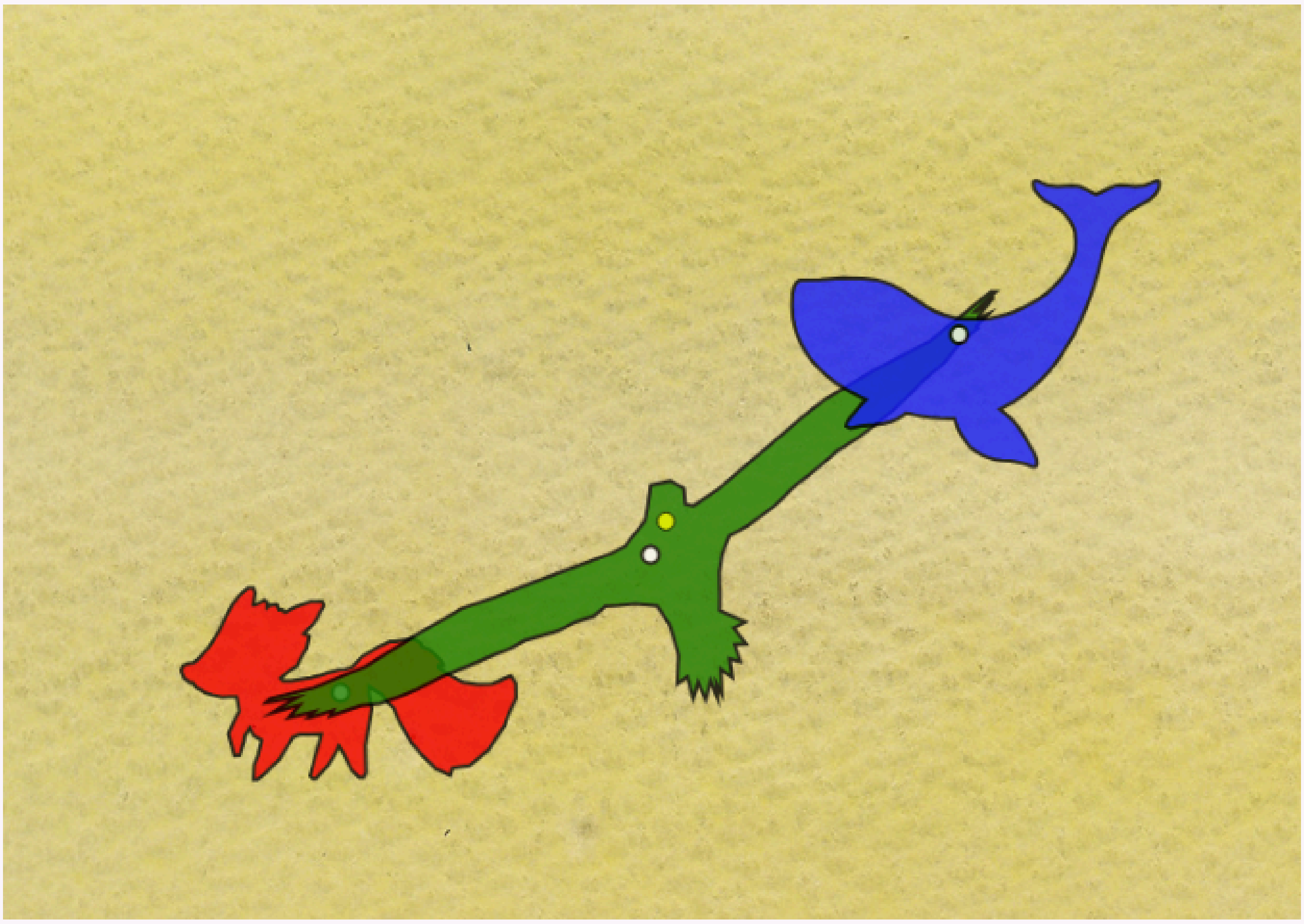
Initial attempts can use a thicker rod for better balance, progressing to thinner rods as children gain experience.

With the Barycenter app children can construct a physical crib mobile, comprising three planar shapes balanced horizontally. Teachers can pre-prepare templates for cutting, and children can assemble the mobile using glue and string. Alternatively, children can design their shapes and save them as a PDF file, which the teacher prints and then the kids can build their own designed crib mobile.

We suggest using the following structure: A lengthy shape, like a bird with extended wings. Two additional shapes can be anything (e.g., different animals). The long shape is suspended from the ceiling by a string. The other two shapes hang from the long shape using strings, separated by a distance (e.g., at the tips of the wings). All three shapes maintain horizontal balance. Use the app to design the shapes and the marked barycenters to find the appropriate locations for string attachments.



Link to the Curriculum

Exploring learning opportunities related to balance and the barycenter concept aids in refining the skill of estimating quantities, areas, and distances and organizing them within systems.

This setting fosters mathematical and process-related competencies like communication, representation, problem-solving, argumentation, and modelling.

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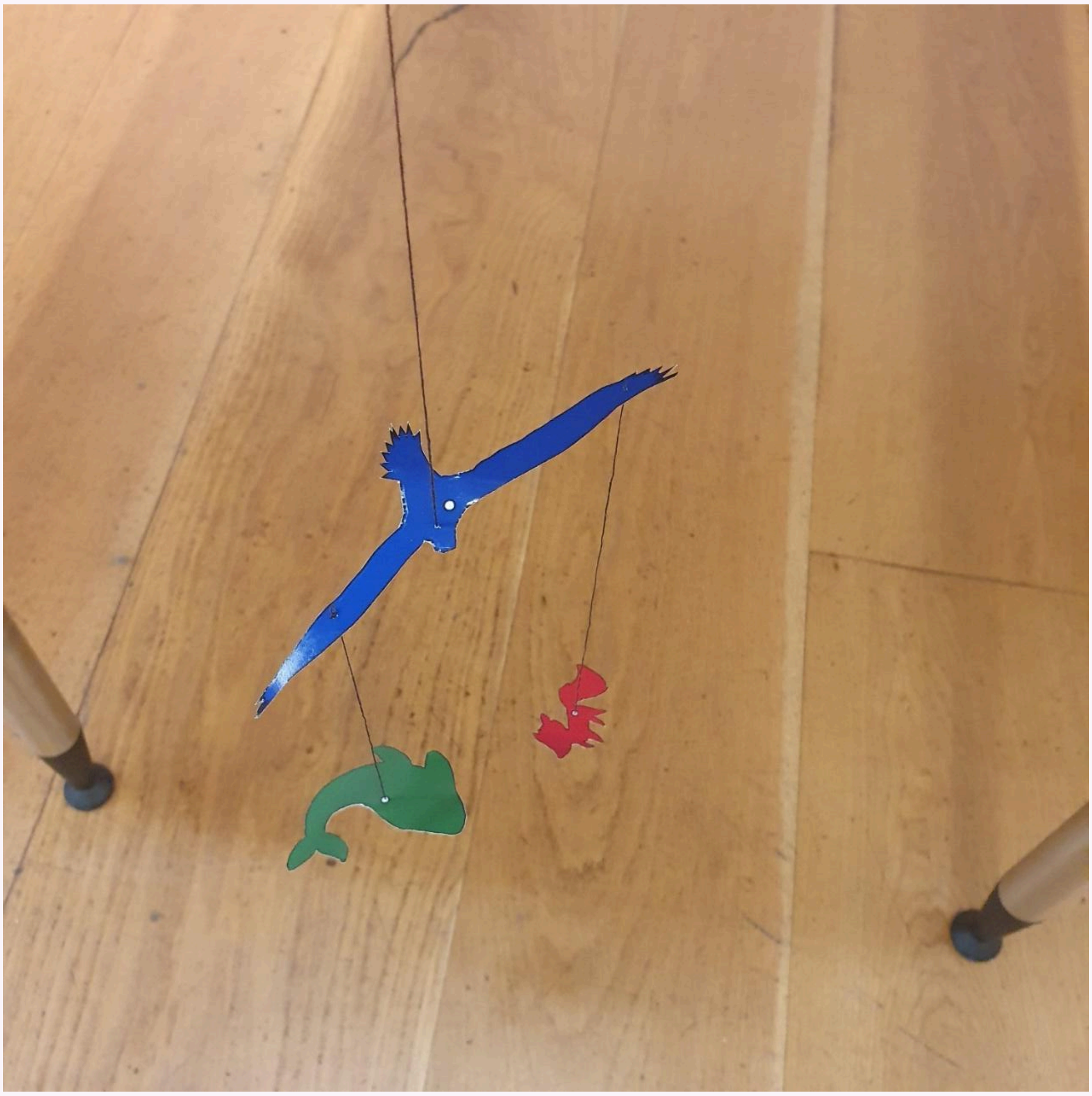
Investigating areas of balance and centre of gravity fosters competencies like accurately comparing areas, shapes, and weights, making and verifying assumptions, and developing a solid understanding of quantities.

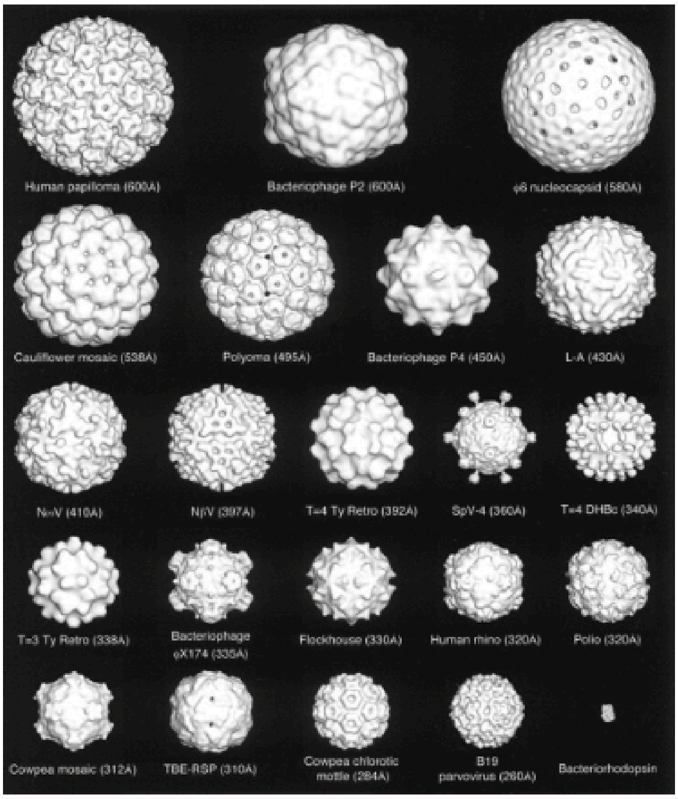
The theoretical background presented at the beginning of this section goes beyond the standard content covered in geometry and mathematics curricula for the target age range of three to eight years old. This content is generally absent from the training of kindergarten or primary school teachers, at least in Germany. However, it offers valuable knowledge for educators, empowering them to decide how to introduce these concepts to children. This early exposure to abstract concepts disguised within games and activities can significantly enhance children's mathematical thinking. Moreover, the content also applies to higher education levels, such as secondary education, where it can introduce essential calculus, limits, vector calculus (e.g., Green's theorem for computing the barycenter), and more.

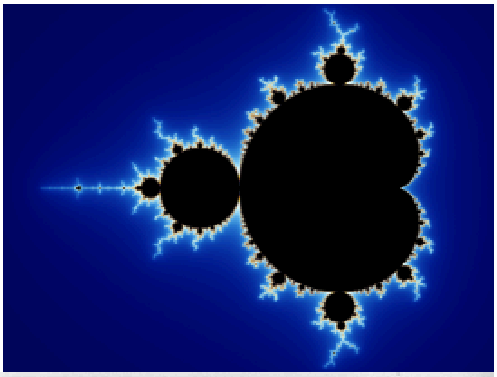


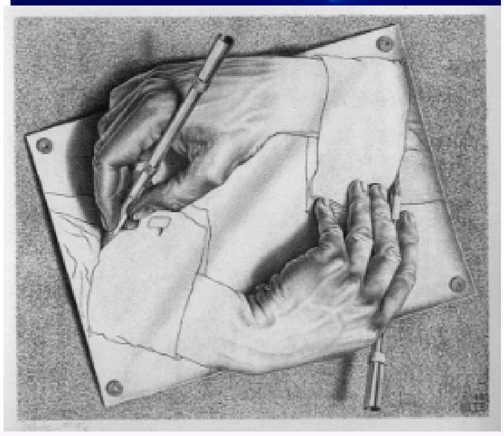
Mirrors and Symmetries

The ideas in symmetry

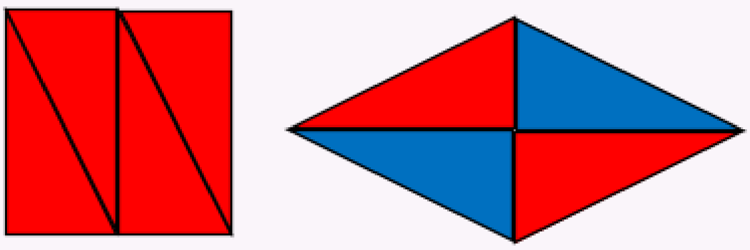
Few concepts can be linked to human experience as much as symmetry. From the most natural and spontaneous discoveries of a small child (the body, the hands, ..., the mirror!) to the various forms of the arts (sculpture, music, architecture, painting), and the sciences (chemistry, physics, biology, and, of course, mathematics).[1]







Here, the most complex elaborations are worth as much as the discovery that an eleven-year-old girl can make when she learns that she cannot transform one polygon into another just with a translation or a rotation but that she must exit the plane to perform a symmetry!



[1] In 2019, in collaboration with the EduCaixa Foundation, the MMACA organised a series of conferences where we developed the concept of symmetry through music, plastic arts, cinema, literature, and museographic language. https://cosmocaixa.org/es/p/espejos-y-simetrias_c379563

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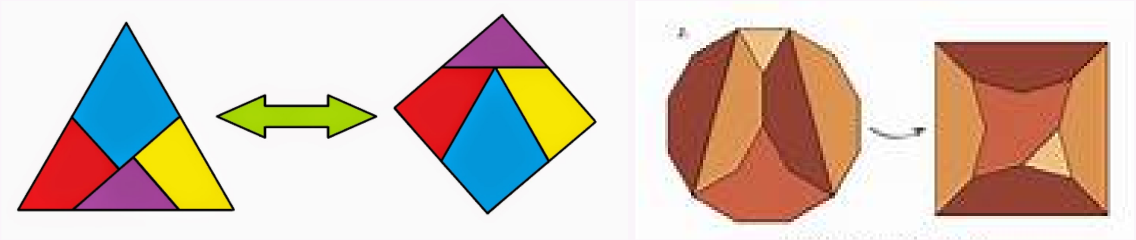
Once discovered, symmetry becomes a part of our mathematical sight, but we have all witnessed how much difficulty it presents for the youngest children to take this little big step and break the bond of the sheet or board and perform a wonderful somersault in space.

Symmetry is so inherent to human thinking that it often represents the primary approach to solving a problem.

A good example concerns the equivalence of polygons through the decomposition and recomposition of their components.

Example

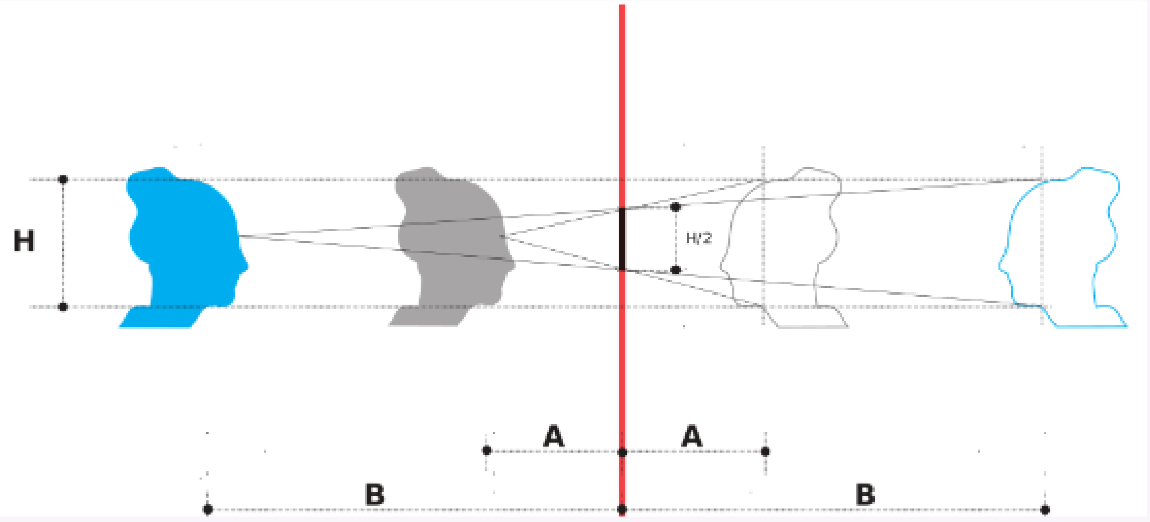
If from a naive, *a priori* vision, it may seem easier to transform a triangle than a dodecagon into a square (and vice versa), in reality, the opposite is true. It is because symmetry guides the transformation of the dodecagon:



When mathematics decided to break away from its role as an arid and abstract discipline and show its more playful aspect, closer to everyday experiences, symmetry, with and without mirrors, obtained a relevant space in the context of museums.

A significant portion of the MateMilano exhibition focused on symmetry, and the same goes for the entire MMACA exhibition in Castelldefels. However, it's noteworthy that symmetry exhibits, mirrors, and kaleidoscopes are typical features in scientific and technological exhibitions across leading museums.

The offer can vary from the disturbing experience of the Mirror Labyrinth at the Tibidabo to making the user contemplate an encounter that seems to challenge conventional perceptions. The exhibits encourage visitors to measure the dimensions of their faces as reflected in an ordinary mirror. Upon closer inspection, they discover that the reflected image represents only half of their face and that its dimensions remain unchanged when the person moves closer or further away from the mirror. What happens?



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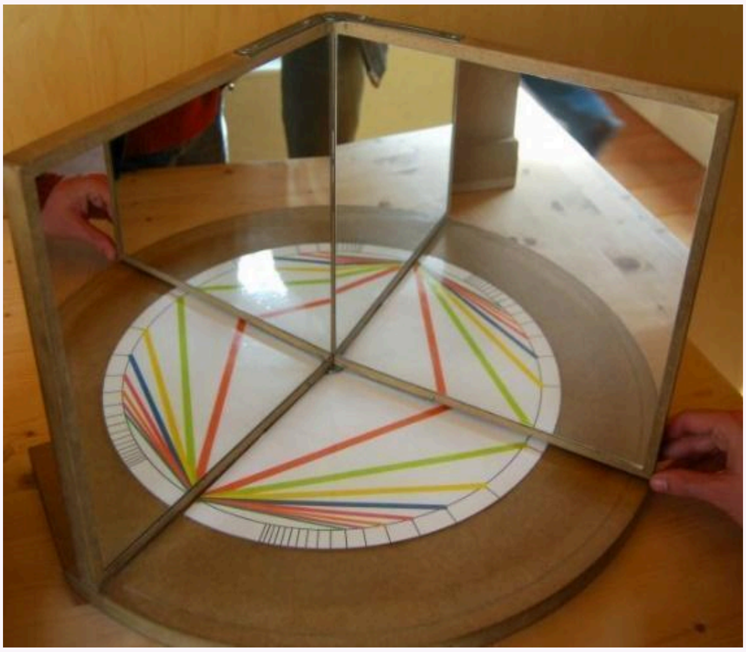
Physics explains the phenomenon, but truly accepting it often requires some reflection (pun intended) and a bit of time spent smearing a mirror at home with the edge of a bar of soap or a blackboard marker.

The following revelation will come through a pair of mirrors connected along one of the sides in the form of a book that will multiply a coin or the number of sides of a polygon as the angle between the mirrors varies.

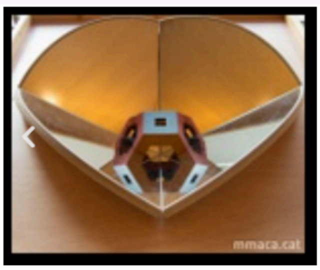


The Book of Mirrors with an inner angle of 90° distinguishes right from left: place yourself exactly in the centre between the mirrors and touch your ear with your right hand. Which hand did your image use? Now, rotate the book while you are still visible in it. Why is your image upside down? And for how many degrees did you turn in the Book of Mirrors? 90º or 180°?

Do we understand that a mirror multiplies by two, two mirrors by four, and three mirrors are placed to form the internal edge of a cube? Let's learn to follow a path reflected in a mirror with our fingers. Forward or backward? How to move your finger if the path curve sways to the left?

Let's learn to draw infinite mosaics between parallel mirrors and not be fooled by the false proportions of Ames room.






We eventually surrender to the allure of kaleidoscopes, where a segment generates the 20 triangles of an icosahedron, or when shifted perpendicularly, the 12 pentagons of a dodecahedron. It's a profound elevation of a simple toy, reminiscent of the transformative impact of Galileo's telescope.

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Link to the Curriculum

Before we begin exploring symmetry-related activities, we need to establish the concept of fitting shapes which has a fundamental role in the curriculum for the first maths cycle. This cycle, which involves children aged 3 to 6, is a crucial period of cognitive development and preparation for more formal mathematical learning. Exploring and manipulating geometric shapes in the early years of kindergarten is essential for laying the groundwork for mathematical understanding.

As a first step, these students are encouraged to manipulate and explore various geometric shapes (such as Pattern Blocks) in a hands-on manner. They learn to recognize these shapes in their everyday surroundings, whether through toys, objects, or even architectural elements. This initial step familiarizes young learners not only with basic shapes like circles, squares, triangles, and rectangles, but also with the composition of these shapes to form others, such as transforming a regular triangle into a rhombus, trapezium, or hexagon.

Next, they are taught to name these shapes, which not only reinforces their vocabulary but above all shows the usefulness of defining and fixing the characteristics of an object, condensing them into a name.

They also learn to differentiate the properties of shapes, such as recognizing regularity and patterns, and compare dimensions: sides, and surfaces. This is a crucial step in developing their ability to communicate and describe shapes accurately.

The assembly of geometric shapes is a crucial pedagogical activity. Students begin to create compositions using these basic shapes, which develops their spatial thinking and creativity. This process is further enriched by the use of mirrors, which double the shapes and stimulate additional cognitive engagement.

In the second phase, the mirror can serve as a tool to divide shapes into smaller units that can be regularly iterated, provided we locate the axis or centre of symmetry.

They represent steps to facilitate a dialogue between two distinct approaches: the analogue, rooted in observation, and the analytical, centred on recognizing variables and developing strategies.

At the outset of the next cycle (ages 6 to 8), students continue their mathematical education by reinforcing the strong foundations established in the first cycle. This initial phase of the second cycle is characterized by a deeper exploration of geometric shapes. More familiar with squares, rectangles, and triangles, students can delve into more complex compositions. They start to assemble intricate figures, treating these basic shapes as pieces of a mathematical puzzle, while addition begins to replace simple counting.

Students continue to explore and identify geometric relationships, thereby deepening their understanding of symmetry and alignment within tangible contexts.

Furthermore, the assembly of geometric figures naturally integrates with other mathematical skills. As students work with flat figures, they develop an understanding of concepts such as perimeter and area. Exploring the concept of the barycenter becomes a fitting next step, as discussed in the chapter on Equilibrium. This process enhances students' overall understanding of mathematics and problem-solving abilities, allowing them to approach problems holistically and translate patterns into formulas.

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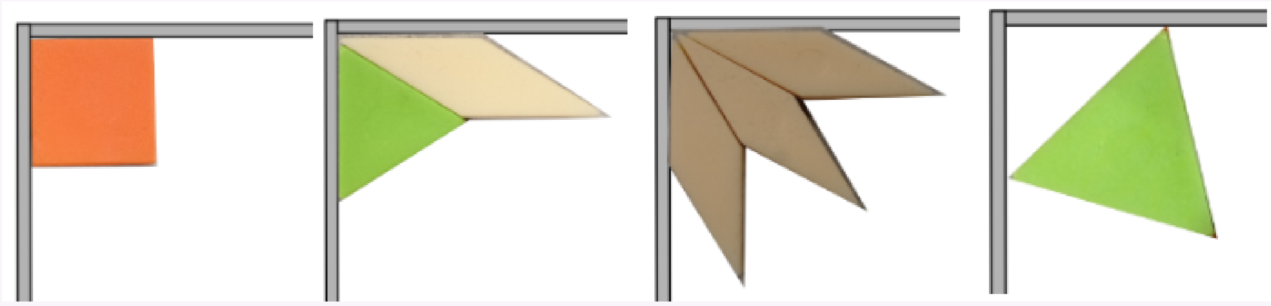
Exhibits from SMEM related to Symmetry

Springing Flowers

You could create different shapes by putting Pattern Blocks facing a book of mirrors with an inner angle of 90°.



Logically, only those shapes that alone or together form right angles will fit between the two mirrors; in other cases, empty areas will appear in the structure. Either way, the built pattern will be symmetrically reproduced in the mirrors.

The activity could entail guiding students through replicating progressively challenging shapes displayed on the board or encouraging them to create original structures on their own. Reflection becomes essential, prompting analysis based on the knowledge acquired at each school stage. It includes consideration of angles, figure composition (highlighting equality of their sides), exploring the relationship between different areas, and so on.

Another advanced skill involves identifying symmetry in given shapes.

An engaging activity with such a powerful emotional impact that you must mediate is taking a selfie with the camera aligned parallel to the face, then duplicating each half of the face with a mirror. This exercise provokes the question: Is our face truly symmetrical?

Yet another investigation could involve studying the symmetry of capital letters.

Here are some examples where Spanish words are reflected in a mirror. An easy task for your students could be to find words in your own language that you could read using a mirror.

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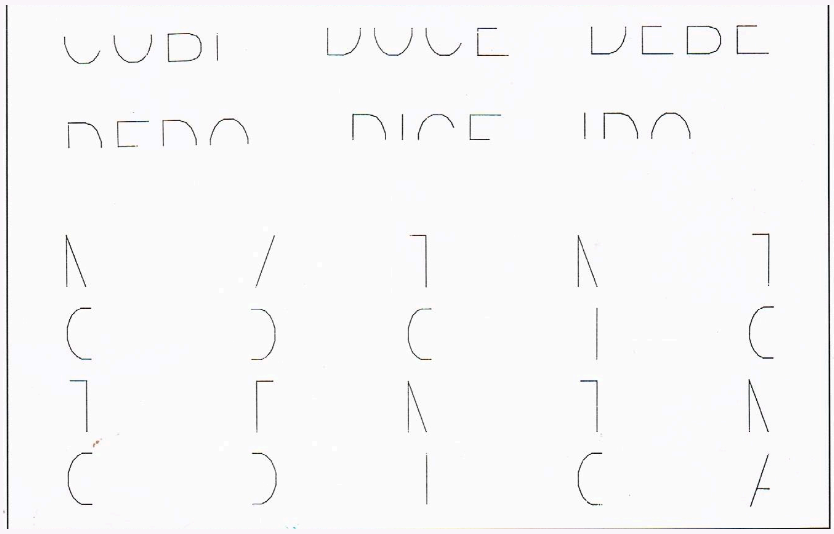
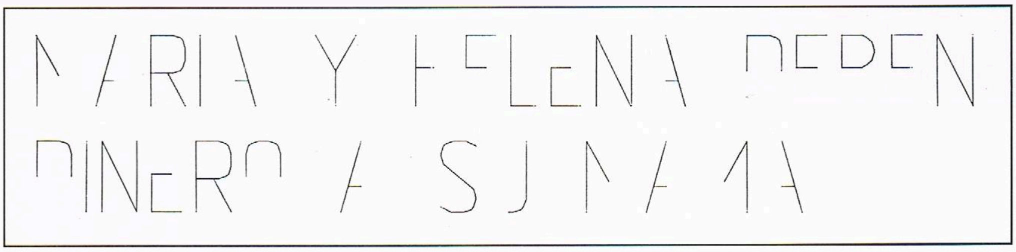
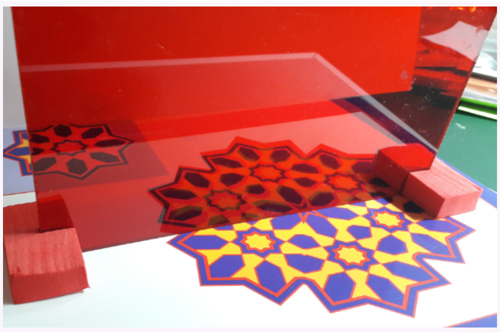


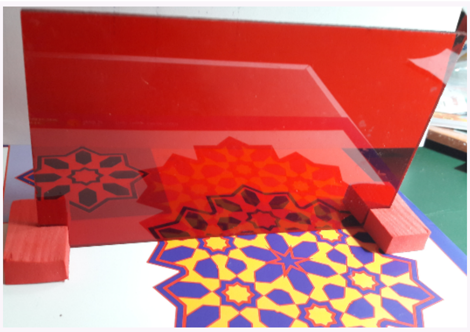
Here are some (Spanish) Secret Messages.



The exploration of symmetry in polygons can lead to engaging activities of increasing complexity. For example, students can search for the minimal portion of a figure that, with the help of one or two mirrors, allows for the reconstruction of the entire shape.

Tools aiding these symmetry exploration activities include the Mira (pictured) and the Georeflector. These are semi-transparent plastic sheets. When you place them on a figure, they reveal the "hidden" half transparently and partially reflected. The discovery of the axis of symmetry occurs when both halves align.





Kaleidoscopes

After exploring the Springing Flowers exhibit, the next step is to progress from two mirrors to three, forming a triangle. This transition leads to the Kaleidoscopes exhibit, available in both physical and virtual versions. Unlike the rosettes ("flowers") observed externally with two mirrors, a three-sided kaleidoscope fills the entire plane with patterns, offering a more immersive experience. However, it can be trickier to see with an angle, so we provide a virtual alternative to the exhibit alongside the physical mirrors.

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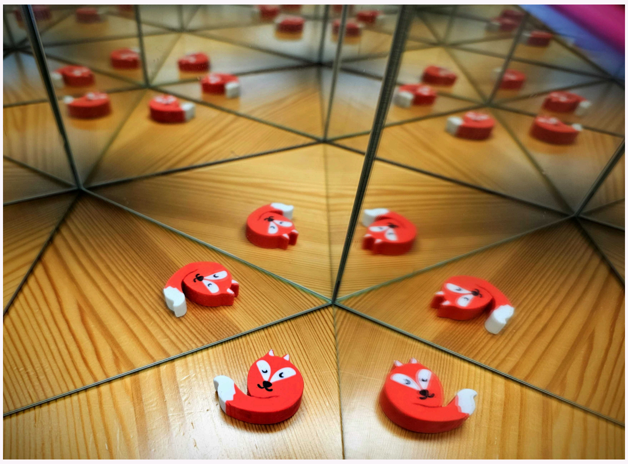
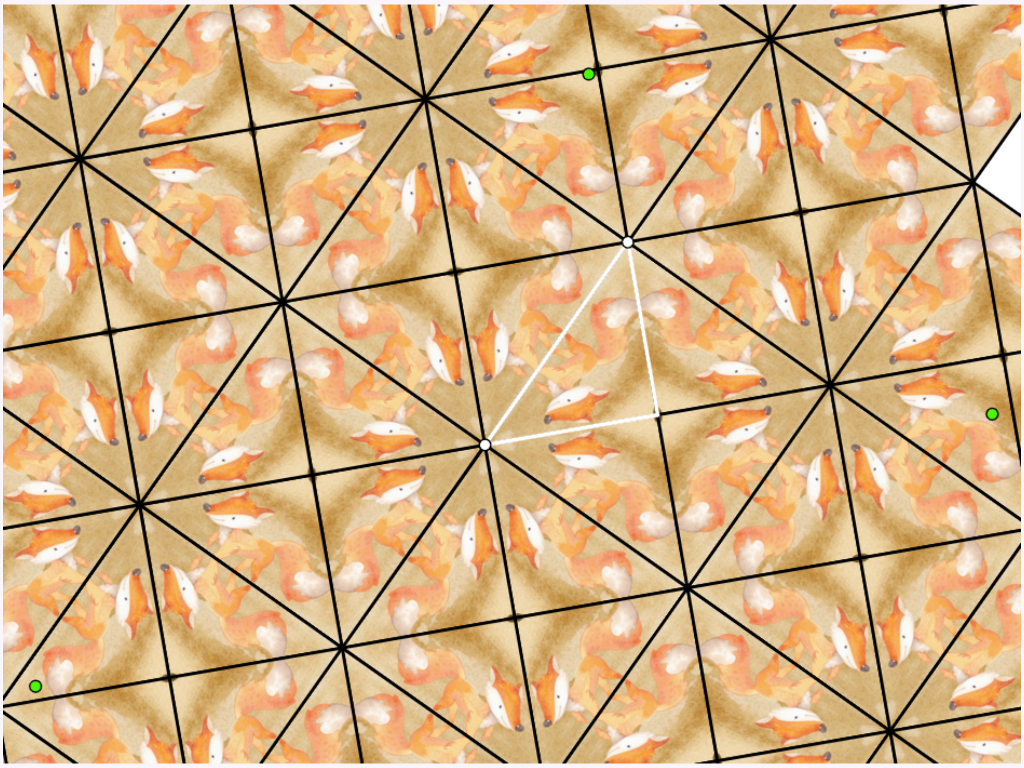
The exhibit features kaleidoscopes fashioned in the shape of special triangles with angles of (60°, 60°, 60°), (90°, 45°, 45°), and (90°, 60°, 30°). When you place an object inside these kaleidoscopes, their reflections fill the plane as seen below:



The photograph above shows a physical setup of the (60°, 60°, 60°) mirror arrangement, forming an equilateral triangle with the three mirrors. The screenshot below is a virtual demonstration depicting an isosceles, right-angled triangle for the (90°, 45°, 45°) kaleidoscope arrangement, similar to the mirror setup in the Springing Flowers exhibit.



The original image of Emy the Fox is marked within the outlined triangle, while all subsequent copies represent mirror images of the original, and mirror images of those mirror images, and mirror images of mirror images of mirror images, and so on. You get the idea.

The virtual exhibit allows you to choose the number of mirror images visible on the screen up to a certain degree. It starts with no mirror images, then shows the rosette of two mirrors, and gradually adds more and more mirror images outside the rosette while maintaining circular symmetry, until the entire visible plane is filled with mirror images.

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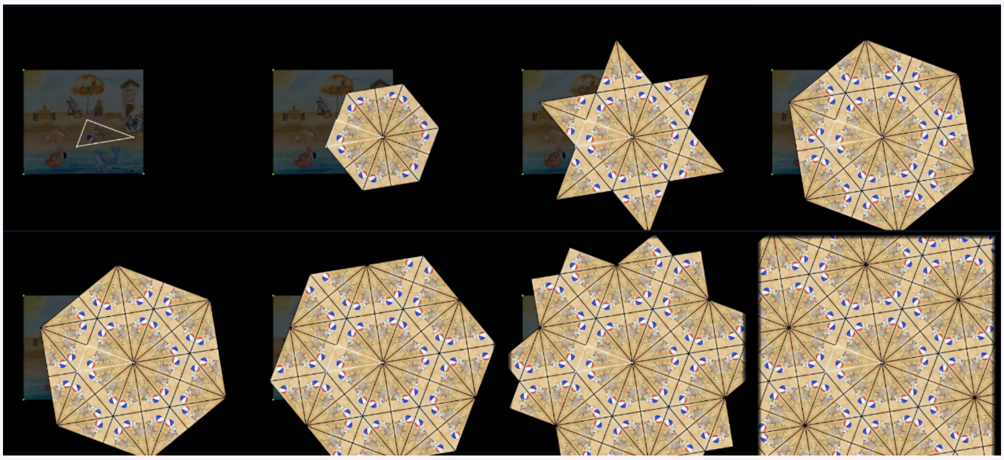
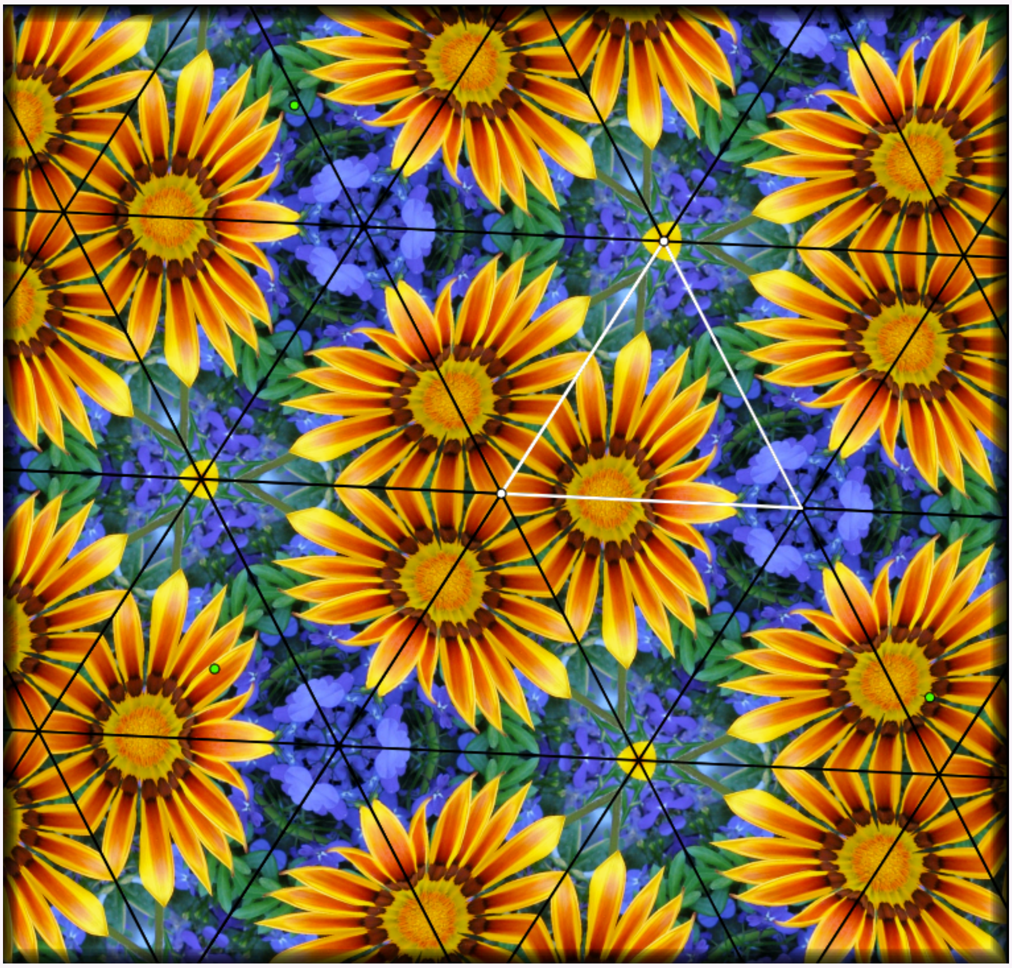


A repetitive pattern covering the plane is known as a periodic tiling or tessellation. It consists of several geometrical shapes, called tiles, arranged without overlaps or gaps to cover the entire plane. In our case, we utilize a single type of tile: a triangle. The angles of the triangle are chosen so that a tiling can be created. You can explore whether other triangular tiles allow for tiling the plane.

A unique periodic tiling emerges when a single regular tile is repeated multiple times, extending infinitely across the plane. A regular tile is one where all sides have the same length and all angles are equal. For example, the simplest regular shape is the equilateral triangle. With the (60°, 60°, 60°) mirror arrangement, a regular periodic tiling is created, showcasing various types of symmetries:



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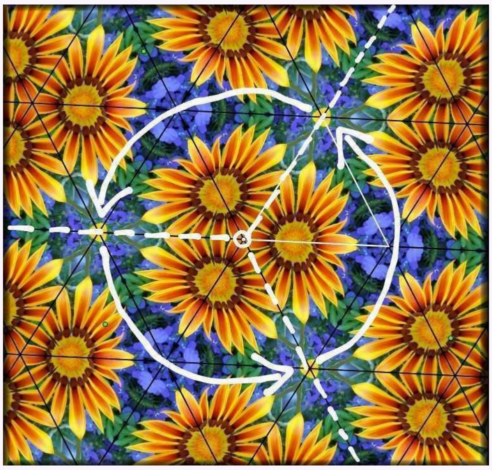
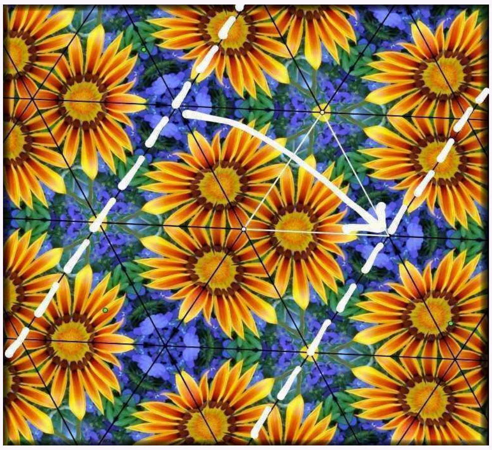
Observing the image, you can identify different types of symmetries:



Rotational symmetry is best understood by selecting any vertex of any of the triangles. Now, keep this point fixed and rotate the pattern around it. You can do this mentally or physically by rotating a printout secured with a pin. If using a physical version, it's important to visualize the pattern extending infinitely across the entire plane, not just on the paper. After a 60° rotation, you'll arrive at a mirrored version of the initial pattern. Another 60° rotation brings you back to the original pattern. This type of symmetry is known as 3-fold symmetry, as the original pattern is reached three times during a complete 360° rotation.



Translational symmetry is achieved by shifting the entire pattern by a specific distance in the same direction, resulting in the same pattern as the original. In the image above, select one of the straight lines—either a horizontal line or one of those diagonally crossing the image (from upper left to lower right, or lower left to upper right). Whichever line you choose, there will be more similar lines, all parallel to each other. Now, envision moving your selected line and placing it on top of the next parallel line. This action reproduces the original pattern. However, if you stop at the next parallel line, you'll arrive at a mirrored version of the original pattern. Try to visualize this process in your mind.

Reflectional symmetry, also known as mirror symmetry, is the most intuitive. You can envision placing a mirror along any of the three types of lines we identified for translational symmetry, resulting in the same pattern.

Then, there's Glide Reflection, which combines translation and reflection. You can imagine moving the entire pattern and then mirroring it (or vice versa), ultimately arriving at the same pattern again.

As an exercise, you can try to identify the different symmetries created by the two other triangular mirror arrangements.

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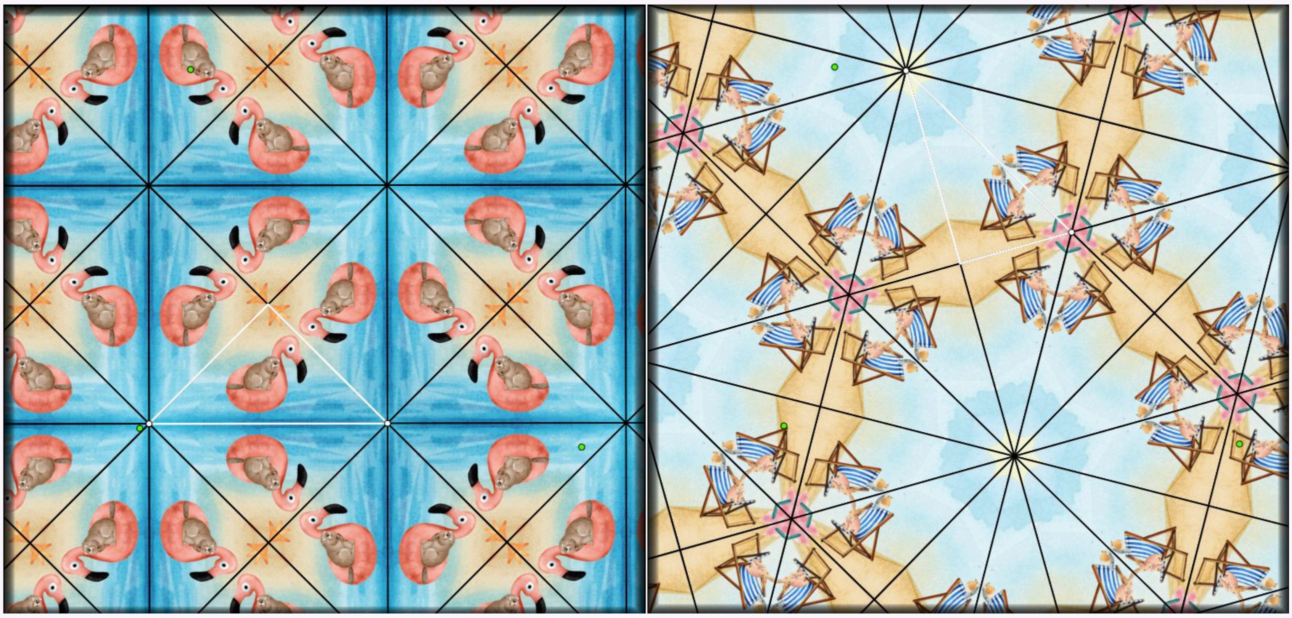

If you're fascinated by the beauty of periodic tiling, you might want to explore the apps iOrnament[2] or Morenaments[3]. These apps allow you to create your own patterns using any of the 17 symmetry groups, which can also be referred to as wallpaper groups. For an introduction to symmetry groups, you can start with the resources available on the open Mathigon platform[4].). Even young children can enjoy creating unique patterns from scratch using these tools

[2] https://www.science-to-touch.com/en/iOrnament.html (available for MacOS only)
[3] https://www.imaginary.org/program/morenaments
[4] https://mathigon.org/course/transformations/symmetry-groups

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Mirror Friends

The original exhibit, tailored by MMACA for pupils aged 6 to 10, is somewhat less complex when compared to the SMEM version. Nevertheless, it demands a fundamental grasp of mental calculation tools and knowledge of addition and multiplication operations (with a focus on factors 2 and 3).

It comprises three boxes labelled with the values 1, 2, and 3, along with a variable number of rubber balls (ranging from 2 to 4) and either one or two dice (depending on the desired difficulty level).

We create a number by placing the balls in the boxes. Rolling the dice determines a number we will make. Each ball's value corresponds to the number displayed on the box which contains the ball.

The first student creates a number, and the second student attempts to replicate it but with a different arrangement of balls.

Here's an example using one die and two balls. If the dice value is 4, the first player can place two balls in the box labelled 2 (2 x 2 = 4), while the other player can put one ball each in boxes 1 and 3 (1 + 3 = 4). This way, both players score one point each.

We can increase the number of possible combinations by adding a third ball.

We can also implement a rule where the correct answer gets as many points as the balls used.

Additionally, we can introduce attractive variations by adjusting the values assigned to the boxes (e.g., using 1, 2, and 4 to simulate the binary system) or by including a "0" (representing the neutral element of addition) and requiring the use of all the balls.

As mentioned, this activity requires basic knowledge of addition and multiplication operations. When we encountered the challenge of adapting it for an audience with limited calculation skills, we found a solution using mirrors.

Mirrors allowed us to simplify the task by shifting from calculation to counting:

- One mirror doubles the objects (1 object + 1 image = multiplication by 2).
- Two mirrors positioned at 120° triple the objects (1 object + 2 images = multiplication by 3).
- Two mirrors positioned at 90° quadruples the objects (1 object + 3 images = multiplication by 4).

This version maintains clarity while breaking down the information into digestible parts.

It may not be easy for everyone to accept that the image is as valuable as the object. We rely on the extraordinary flexibility of children's imagination to ensure full acceptance of these rules.

We believe that a virtual version of the module, where all objects displayed on the screen, whether introduced or appearing spontaneously, hold the same virtual value, is even easier to accept. This approach generates, at least to some extent, the same level of engagement and understanding.

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Drawing Dice

The original idea stemmed from a children's toy, where the goal was to replicate faces printed on cards.

Inspired by this concept, we decided to apply the same principle but with geometric shapes, leveraging the symmetry inherent in these shapes and cubes. As a result, we created four wooden dice, each side marked with different shapes in black. All the dice are identical in size and contain the same six faces.

You could create the dice by using origami or cardboard. Assemble it with glue and paint the sides. While different drawings can replace the existing ones to introduce the possibility of constructing new shapes, the ones we selected are specifically chosen for their simplicity (being all corner-like) and suitability for educational activities.

Upon receiving these cubes, the first instinct is to playfully explore the shapes they can create: circles, two different triangles, two different squares, a four-pointed star, an ice cream cone, and more. Students must rotate and manipulate the cubes to form these shapes, soon discovering that only two faces are asymmetric, necessitating their use in pairs to achieve overall symmetry.

After this initial exploration, guided activities can be introduced:

- Make all possible polygonal shapes with 3 or 4 sides.
- Make all possible symmetric shapes (there are many if we accept the ones with rotational symmetry!)
- Classify some of these shapes by area (without computing it!)

After the initial exploration and play, you can introduce various classroom activities, guiding students with appropriate questions to help them develop their understanding:

- What is the area of the figure formed, and how do you calculate it?
- What is the perimeter of the figure, and how do you calculate it?
- Compare the perimeter and area of proportional squares.
- Create new shapes and calculate their area and perimeter.

It is crucial to explore the intrinsic properties of geometric figures and leverage symmetry to calculate areas and perimeters without relying on formulas. This approach fosters the development of students' problem-solving skills.

As an additional activity, you could ask students to invent their own dice, draw shapes on each face, and design a unique set of dice using origami or 3D printing.

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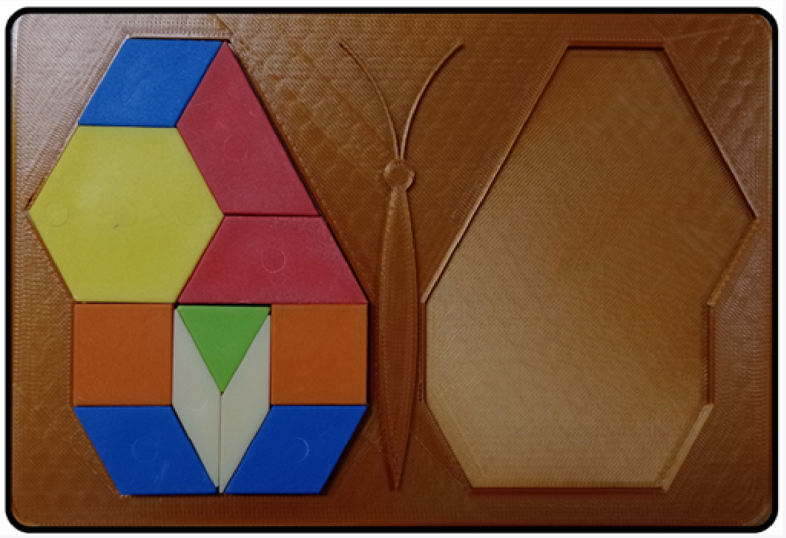
Make Me Wings

This activity emphasizes the fusion of creativity and problem-solving, just like the rest of SMEM exhibits.



The first step involves replicating a composition of elementary shapes (Pattern Blocks) on one wing of a butterfly silhouette.

This task requires identifying the pieces (triangles, squares, rhombuses, trapezoids, and hexagons) and arranging them symmetrically within the empty shape. The challenge may not be immediately straightforward, especially for young children.

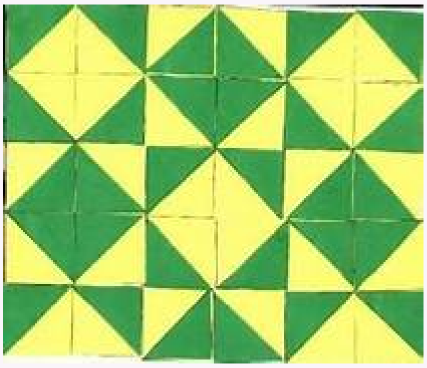
In the second part of the activity, participants compose both wings using the provided shapes. You could provide the essential pieces to guide, though not enforce, the construction of symmetrical wings. Alternatively, you could supply additional pieces to allow for more freedom in wing design.

One of the appealing aspects of this activity is its adaptability. You could easily tailor it to suit individual students, allowing each to design their own butterfly wings.

Examples of activities with the same material

In this chapter, let's explore a few additional activities you could conduct using the same educational materials.

Some mosaics, such as the classic two-tone tile, are easy to build. Different combinations can yield engaging results, especially when using a mirror. Alternatively, placing the mosaic between two parallel mirrors can create aesthetically pleasing effects.



The Chinese tangram also offers opportunities for exploration in the realm of symmetry.

Another intriguing activity is the construction of the Nasrid Bone, which serves as a foundation for various variations on the theme. Starting with regular or semi-regular figures that tessellate the plane, students can delve into this construction.

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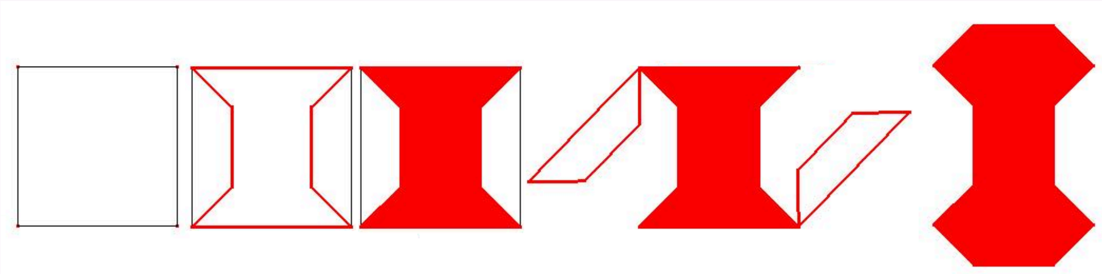
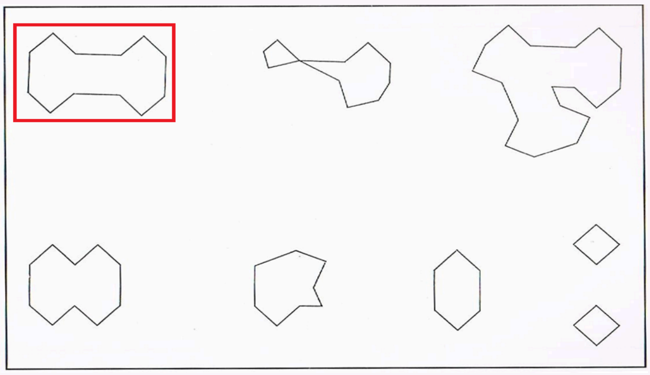
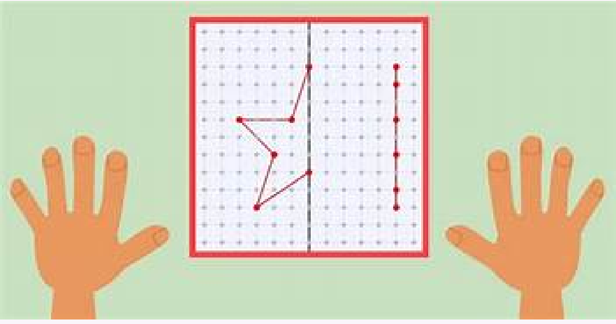




Building upon the Nasrid Bone concept and incorporating a mirror, you could introduce a simple, yet captivating activity by Rafael Pérez. This activity encourages students to create and recognize symmetries.

Additionally, the Geoboard offers numerous training activities for constructing geometric figures.



Conclusion

We believe that symmetry-related activities represent a good example to test the hypothesis that there is no small mathematics or small mathematicians. In other words, well-designed activities intended for younger learners inherently contain the foundational elements of mathematical thinking. They could become significant even for older learners with advanced skills.

It is a discussion that began a few years ago, in an edition of the Matrix Conference and which involved many of the partners of the SMEM project. The discussion allowed us to compare proposals and experiences. This new context enriches our wealth of educational offers to be addressed in the coming months to teachers in each partner's area of influence.

We think that the following experience presents all these elements: a targeted and motivating context and language, an analogical and analytical approach, varying degrees of difficulty, utilization of strategies, encouragement of creativity, and the acquisition of diverse skills.

It is worth noting that although our encounter with this material was coincidental, the resource quickly revealed its adaptability to our dual target audience: children aged three to eight and their teachers.

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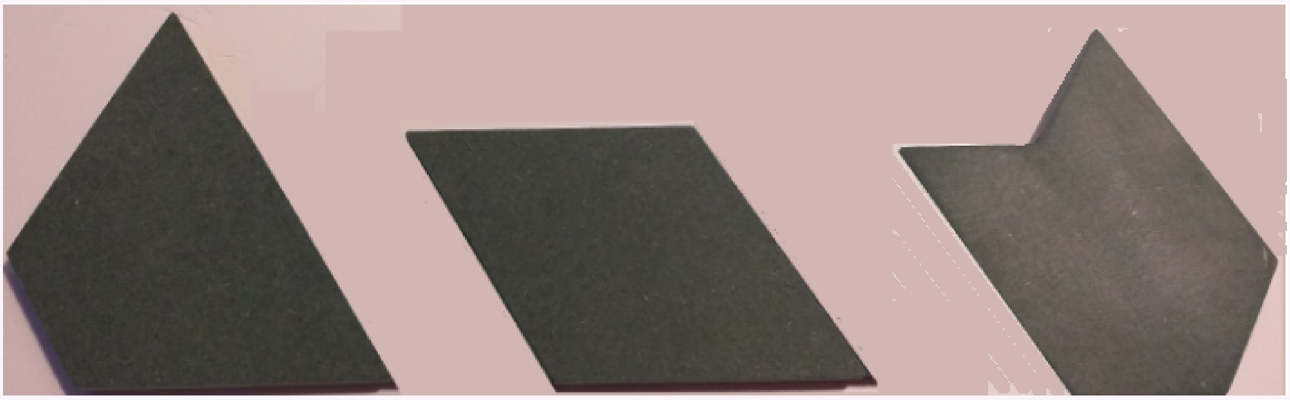


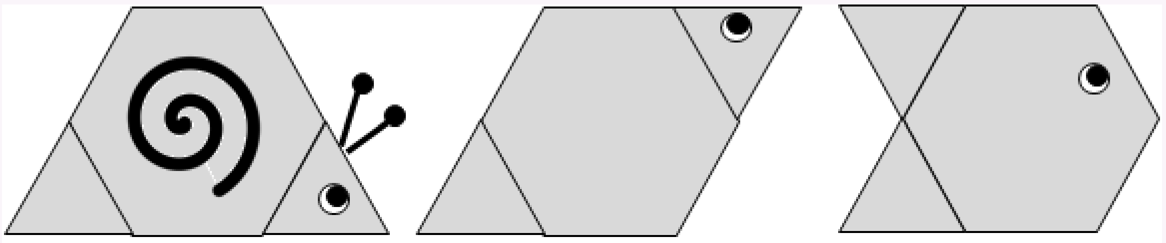
The original proposal, a symmetry puzzle called Baikonur, by Alexander Magyarics, challenged participants to assemble three pieces to form a single shape with an axis of symmetry

While this challenge may be too difficult for young children, the forms are suggestive, and you might start with less complex activities.

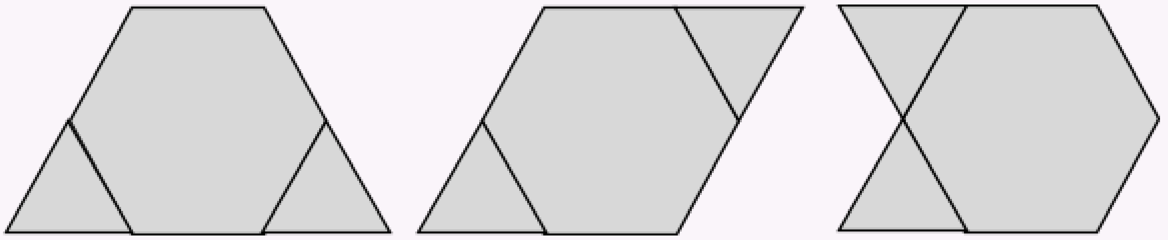
Emy's New Symmetrical Adventure (with Tasks)

Emy wishes to visit her friend Heidi the Whale.

Her friends are eager to help her—meet Sam the Snail, Maria the Marmot, and François the Fish.



Task 1: Find the symmetry axis of each figure (a mirror might be of help here).



The three friends form three pairs to work together, on different days, to find a way to make Emy travel to meet Heidi in the sea.

Sam and Maria design a boat, but it is too fragile for mighty sea waves.

Sam and François project a canoe, but it is too small for Emy the Fox to fit inside.

Maria and François propose a rocket, but it is too noisy.

Task 2: Find the line symmetry of the shapes obtained by joining the pieces two (out of three) by two (they must always have at least one side (or part of it) in common).

So, they conclude that all three should work together.

They decide to build a sailboat to be able to reach Heidi into the sea.

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But even if François, the Fish, should know the secrets of the liquid environment, the friends are not skilled marine carpenters, and the boat leaks!

Even if the hole is not at the bottom of the hull, they know that the waves will make the boat take on water and sink!!

Challenge 3.1 Combining the three shapes, build a silhouette of the sailboat, symmetrical and in compliance with the rules (one side or part of it in common).

Realizing their poor shipbuilding skills, the friends settled on constructing a simpler boat.

«How about if we build a canoe? » - Sam suggests.

«Yes, but bigger than the one François and you designed» - said Maria.

So, they made it.

Challenge 3.2 Combining the three shapes, build the silhouette of a canoe, symmetrical and in compliance with the rules (one side or part of it in common).

And they were ready to show it to Emy.

Hey, but… where's Emy?

Challenge 3.3 Combining the three shapes, build the silhouette of Emy, symmetrical and in compliance with the rules, quite like the icon of the SMEM project (but only with an empty triangle in).

And this was how Emy managed to travel across the sea and find Heidi.

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Fitting shapes

Definition of Fitting shapes

Fitting shapes in kindergarten is a fun and educational activity that allows young children to develop their understanding of shapes, basic geometry, and problem-solving skills.

Here are some ideas for geometric shape assembly activities for kindergarten pupils:

- Shape Puzzles: Provide simple puzzles with pieces of different geometric shapes. Children should match the pieces to form a picture or a complete shape.

Le Kéor - Credits Fermat Science

- Building with Blocks: Use building blocks of different shapes (squares, triangles, circles, etc.) and encourage children to create structures using these shapes.

Fractionary- Credits Fermat Science

- Shape Collages: Give children pieces of paper in various shapes (circles, squares, rectangles, triangles) and different colours. They can create images or patterns by glueing these shapes onto a sheet of paper.

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- Tangram Games: Tangrams are puzzles composed of seven different geometric shapes. Children can manipulate them to form various figures and develop their understanding of shapes.

Tangram - Credits OpenClipart

- Creating Characters: Encourage children to create characters using geometric shapes for the body, eyes, nose, etc. They can invent stories featuring their creations.
- Shape Hunt: During an outdoor walk, ask children to spot objects with specific shapes, such as circles (car wheels), rectangles (house windows), etc.

Matermaths - Credits Fermat Science

- Shapes in Nature: Explore nature with pupils and look for examples of geometric shapes in the world around them, such as triangular leaves or round rocks.

These activities allow children to have fun while developing their understanding of shapes and geometry, which makes a base for their future mathematical education.

Link to the Curriculum

The concept of fitting shapes plays a fundamental role in the curriculum for the first maths cycle. This cycle, which involves children aged 3 to 6, covers a crucial period of cognitive development and preparation for more formal mathematical learning. Exploring and manipulating geometric shapes in the early years is essential for laying the groundwork for mathematical understanding.

First, these pupils are encouraged to manipulate and explore diverse geometric shapes using physical materials. They learn to identify these shapes in their everyday environment through toys, objects, or even architectural elements. This initial step familiarises young learners with basic shapes such as circles, squares, triangles, and rectangles.

Next, they learn to name these shapes, thus reinforcing their mathematical vocabulary. They also acquire skills in differentiating the properties of shapes, such as recognizing equal sides in a square or right angles in a rectangle. It is a vital step in developing their ability to communicate and describe shapes accurately.

The building of geometric shapes is an essential pedagogical activity. Pupils begin to create arrangements using these basic shapes, which develops their spatial thinking and creativity. It also prepares them for later understanding of more advanced concepts such as symmetry, alignment, and even the introduction to three-dimensional solids.

In summary, having the concept of fitting shapes in the curriculum for the first cycle of mathematics is a crucial step in establishing a solid foundation in geometry and preparing pupils for more advanced mathematical concepts as they progress in their educational journey. It promotes the development of mathematical language, spatial thinking, and creativity while giving young learners a positive initial experience with mathematics.

For the beginning of the next cycle (ages six to eight), pupils continue their mathematical learning by consolidating the solid foundations earned in the first cycle. The second cycle begins with a thorough exploration of geometric shapes. Students, now more familiar with squares, rectangles, and triangles, can go further. They can dive into the assembly of these basic shapes, tackling the challenge of mathematical puzzles to construct complex figures.

With the introduction of three-dimensional shapes, better known as geometric solids, students discover the third dimension. They learn to create cubes, cylinders, prisms, and other solids that enable us to produce fascinating three-dimensional structures. Through these activities, students not only comprehend the concept of solids but also enhance their spatial abilities by envisioning the integration of these shapes to construct more intricate objects.

The assembly of geometric figures goes beyond simple piece manipulation. It also serves as a learning ground for symmetry and alignment, essential geometric concepts. Pupils continue to explore and identify geometric relationships, thus strengthening their understanding of symmetry and alignment in concrete contexts.

Finally, the assembly of geometric figures naturally connects to other mathematical skills. Students begin to comprehend the concepts of perimeter and area when working with flat figures, enhancing their overall understanding of mathematics and their ability to solve problems holistically.

Exhibits from SMEM Project related to this concept

The use of mathematical exhibition modules that explore the concept of fitting shapes is an exciting opportunity to awaken learners' curiosity and immerse them in the fascinating world of mathematics. These exhibits aim to introduce pupils, from the early years of their educational journey, to a series of key concepts and skills related to geometry and spatial thinking.

Here is a list of 11 modules that you can find in the open-source SMEM project:

Module 1 Forest Puzzle
Module 2 Cherry Pies
Module 3 9 Foxes
Module 4 The Beaver's Dam
Module 5 Cubing

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Module 6 Drawing dice
Module 7 Animal Houses
Module 8 Building Bridges
Module 9 Coloured Wings
Module 10 Make Me Wings
Module 11 Happy Neighbours

Some possible exhibits connections

Example 1: Shape Reproduction

We propose designing an interesting pedagogical sequence focused on the theme of Shape Reproduction by combining some of these exhibition modules. To develop this, we will rely on three exhibits: Drawing Dice, Make Me Wings, and Coloured Wings. These modules offer an engaging perspective on the process of shape reproduction. During this sequence, pupils will have the opportunity to explore the concepts of symmetry, patterns, and repetition while developing their observation and creativity skills. By encouraging them to create their own works inspired by these exhibits, we promote individual expression while exploring fundamental concepts related to shape reproduction.

Example 2: Construction/ Spatial Awareness

Next, we could design a pedagogical sequence focused on Construction/Spatial Awareness. We can use the following exhibition modules: Beaver's Dam, Cubing, Animal Houses, and Building Bridges. These exhibits offer a multidimensional approach to exploring construction and spatial awareness. During this sequence, pupils have the opportunity to develop problem-solving skills, geometry skills, spatial understanding, and collaboration.

Example 3: Mathematical Logic

We have a great opportunity with this project to design a stimulating pedagogical sequence around Mathematical Logic, using original exhibits such as Forest Puzzle, 9 Foxes, and Happy Neighbours. These exhibits offer rich perspectives for exploring mathematical logic in various forms. During this sequence, pupils can develop their logical thinking, problem-solving, and mathematical skills while having fun. By solving mathematical puzzles with Happy Neighbours, exploring the mysteries of 9 Foxes, and solving the Forest Puzzle, pupils could apply complex mathematical concepts through practical application.

Example 4: Number Composition

Finally, we can design a stimulating pedagogical sequence around Number composition using the exhibit Cherry Pie. This exhibit offers a visual and playful approach to exploring number composition in depth.

Examples of activities with the same material

In this chapter, we will explore several examples of additional activities that can be done using the same educational material, namely geometric shapes. These activities offer a variety of

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opportunities to reinforce the understanding of mathematical concepts while stimulating pupils' engagement.

Geometric Shapes Sudoku

Geometric Shapes Sudoku is a creative and challenging version of traditional Sudoku. Instead of using numbers, pupils use geometric shapes to complete the grid. The goal is to place each shape in the grid so that no shape repeats in the same row, column, or block. This activity enhances problem-solving, logic, and shape understanding. Children must analyse the spatial relationships between shapes to succeed.

Tangram: Shape Exploration

Tangram is a set of seven geometric shapes that can be assembled to create a wide variety of figures. Pupils can explore the properties of the pieces, compare them, and combine them to create complex shapes. This encourages understanding of shapes, geometric transformations, and symmetry concepts. The children also develop their spatial thinking by visualising how the pieces fit to form different figures.

Tiling: Pattern Repetition

Tiling activity involves using geometric shapes to create repetitive patterns on a flat surface. Pupils can explore how shapes fit together to cover a surface without leaving gaps or overlaps. This reinforces their understanding of patterns, transformations, and tiling concepts. They can create artistic patterns or complex mathematical tiling.

Frieze: Creating Repetitive Patterns

A frieze is a sequence of repetitive patterns that can be used to decorate borders or surfaces. Pupils can use geometric shapes to create friezes by repeating a pattern or sequence of patterns. This activity promotes creativity and understanding of repetitive patterns. They can also explore symmetry concepts in creating their friezes.

Free Construction

Allowing pupils to explore free construction with geometric shapes is an excellent way to stimulate their creativity and reinforce their understanding of geometric concepts. They can create patterns, sculptures, buildings, and more using shapes as building blocks. This activity encourages spatial thinking, problem-solving, and the discovery of geometric properties through hands-on experience. Pupils can also collaborate to build larger and more complex structures, strengthening their communication and teamwork skills.

These examples of additional activities illustrate the versatility of educational material based on geometric shapes. By integrating these activities into your teaching, you can offer students a range of stimulating learning experiences that reinforce their understanding of mathematical concepts while promoting creativity and critical thinking. Using concrete material like geometric shapes allows pupils to explore mathematics in a practical and engaging way, which enhances their enthusiasm for learning mathematics.

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Free construction - Crédits Fermat Science

Conclusion

In conclusion, fitting shapes in kindergarten is a valuable educational approach for the development of young children. This theme offers many engaging activities that promote understanding of shapes, spatial thinking, creativity, problem-solving, and preparation for more advanced mathematical concepts.

The examples of activities presented in this chapter demonstrate the richness and diversity of learning experiences that can be offered to pupils using the same educational material, such as geometric shapes. Fitting this theme into the curriculum aligns perfectly with the educational objectives of developing mathematical skills from the early years of formal education.

The exhibits offered in the open-source SMEM project provide a solid foundation for creating coherent and enriching pedagogical sequences. These modules promote exploration, discovery, and practical application of mathematical concepts while stimulating pupils' curiosity. The possible connections between modules open the door to an interdisciplinary approach to learning, where students can explore mathematical concepts while developing skills in other areas, such as problem-solving, critical thinking, communication, and creativity. Additional complementary activities, such as Geometric Shapes Sudoku, Tangram, tiling, creating friezes, and free construction, provide even richer learning opportunities.

Finally, fitting shapes in kindergarten is not just a fun activity. It is a solid foundation for preparing pupils for their mathematical journey. It promotes the construction of mathematical knowledge while igniting a passion for mathematics in young learners. This theme contributes to creating a stimulating and fulfilling educational environment where pupils can develop their understanding of the world around them through the lens of mathematics. By investing in this innovative pedagogical approach, we contribute to shaping a new generation of passionate and competent mathematics learners ready to face the challenges of tomorrow.

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Observation and Counting

Mathematical Concepts of Observation and Counting for Young Children

Mathematics is a language that surrounds us, even from the earliest stages of our lives. A fact that directly opposes the common belief that a person is born with a talent for mathematics or not. Mathematics is an important life skill that can be acquired and honed through learning, dispelling the notion that one needs innate talent to excel in this field. Every individual has the potential to grasp mathematical concepts and develop proficiency, regardless of initial aptitude. The key lies in tailored and personalised approaches to learning, aligning instruction with individual interests and current knowledge levels. By recognizing that mathematics is a learned skill, we empower learners to embrace the subject with confidence, fostering a growth mindset that encourages exploration, curiosity, and continuous improvement. This inclusive approach ensures that mathematics becomes an accessible and enjoyable journey for everyone, promoting the idea that math is not a talent but a skill cultivated through effort, practice, and the proper educational support.

For children between 3 and 8 years of age, the foundations of mathematical thinking are laid through counting and observation.

Counting provides a tangible entry point into the world of mathematics by enabling kids to comprehend and manipulate numbers. Through counting, children begin to recognize numerical patterns, develop an intuitive sense of quantity, and understand concepts like addition and subtraction. Moreover, counting enhances their observation skills as they discern differences and similarities between objects. In that way, counting paves the way for more complex mathematical reasoning later on. It instils a sense of order and organisation, which are vital mathematical principles. Counting not only equips children with a practical tool for everyday problem-solving; it also nurtures their mathematical curiosity and confidence, laying the groundwork for a lifelong adventure of mathematical exploration and discovery.

Observation isn't just about seeing; it's about noticing details, patterns, and relationships. This skill is fundamental not only for mathematical thinking but for a child's overall cognitive development, as it cultivates their ability to discern patterns, relationships, and details in the world around them. Through keen observation, children learn to identify shapes, sizes, colours, and spatial arrangements, all of which are fundamental mathematical concepts. Additionally, observing the natural world, objects, and even everyday routines allows them to grasp concepts like symmetry, sequencing, and measurement. It encourages them to ask questions, make hypotheses, and draw conclusions—a process akin to the scientific method, which

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underpins mathematical inquiry. This process of keen observation not only sparks mathematical curiosity but also nurtures critical thinking skills that are vital for problem-solving and mathematical reasoning. In essence, observation becomes the lens through which young learners perceive and engage with mathematical concepts, serving as the bedrock upon which their mathematical understanding is constructed.

Before we move on, here are some real-life examples readily available to parents and teachers for practicing concepts of observation and counting.

Making a Bracelet from Wooden Beads: Craft activities provide excellent opportunities for counting and pattern recognition. When creating a bracelet, ask the child to choose beads of different colours and sizes. They can practice counting as they thread each bead onto the string and create patterns by arranging beads in a sequence.

Nature Scavenger Hunt: Take a nature walk in a park or your backyard and create a scavenger hunt list with items to observe and count. For instance, "Find three different types of leaves" or "Count how many birds you see." This activity encourages observation skills and numerical awareness.

Searching for Sea Shells on the Beach: Take a stroll along the shoreline with your child and engage them in counting and observation. How many different types of shells can you find? What patterns or shapes do you notice? Counting these treasures can turn a beach walk into a mathematical adventure.

Counting the Stars: On a clear night, lay out a blanket and gaze at the stars with your child. Count the stars you can see and encourage them to spot constellations. This activity fosters both counting skills and the ability to observe patterns in the night sky.

Grocery Shopping: While grocery shopping, involve your child by asking them to count items you put in the cart. For instance, "Let's put three apples in the cart" or "We need six eggs." This simple activity reinforces counting in a real-world context.

Cooking Together: Cooking offers various opportunities for counting and observation. Ask your child to count the number of ingredients needed for a recipe, such as cups of flour or teaspoons of sugar. They can also observe how ingredients change during the cooking process.

These examples demonstrate how observation and counting can be seamlessly integrated into everyday activities, enriching a child's mathematical understanding while fostering a sense of wonder about the world around them. In the following sections, we will delve into specific activities and workshops inspired by the SMEM project, designed to make mathematics an engaging and joyful experience for young children.

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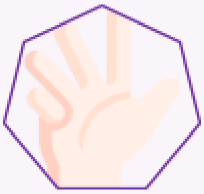
Link to the Curriculum

Good maths education for young kids means matching teaching methods with curriculum goals. The SMEM project provides a valuable framework for integrating mathematical concepts of observation and counting into the curriculum for children aged 3 to 8.

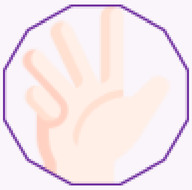
In kindergarten, children are introduced to the world of mathematics through play-based and exploratory activities. Counting and observation are fundamental to this stage of learning.



The kindergarten curriculum typically includes basic counting skills, where children learn to count from 1 to 10 and beyond. Counting is integrated into daily routines, such as counting the number of children present, counting objects during playtime, or counting steps during a nature walk. These activities not only develop numerical awareness but also enhance language skills.

Observation in kindergarten involves helping children notice details in their surroundings. It includes identifying shapes in everyday objects, recognizing patterns in their clothing or the classroom, or observing how objects change in size, colour, or position. These observations lay the foundation for pattern recognition and critical thinking.

As children progress to primary school, mathematical concepts become more structured and comprehensive. Counting and observation skills continue to play a vital role in the curriculum.



In the primary school curriculum, counting evolves into more complex tasks, including addition and subtraction. Students not only count objects but also learn to add and subtract numbers within specific ranges. Counting becomes a tool for solving real-life problems, such as calculating the total cost of items at a store or sharing objects equally among classmates.

Observation skills in primary school extend beyond recognizing patterns in objects. Students are encouraged to observe and interpret data, charts, and graphs. They learn to analyse information critically, make predictions, and draw conclusions. This form of observation is crucial for understanding concepts like data representation and statistics.

Exhibits from the SMEM Project related to Counting and Observation

Representing numbers

In the Representing numbers exhibit, learners should match the tokens with forest-themed pictures to the numbers 1-10 found on the board. One option to extend this activity in the classroom is to use tessellations. Tessellations are 2D geometrical patterns that fit together without gaps or overlaps and can repeat in all directions for infinity. In the pictures below, you can see examples of tessellations using regular polygons.

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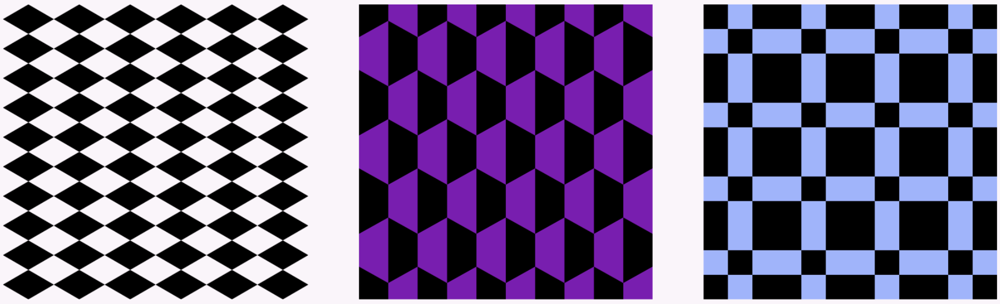
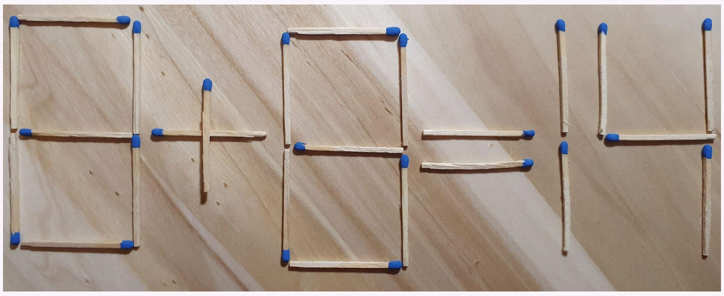


Examples of tessellations using polygons (rhombuses, trapezoids/ hexagons, squares)

One activity to use tessellations in combination with counting is to allow learners to select 1-4 shapes that can be used as patterns. The teacher can also play the find the difference game to see which shapes do not belong in the pattern.

These activities could be extended during nature walks when kids could be challenged to create beautiful patterns by using natural elements such as rocks, leaves, berries, flowers, seashells, etc.

A different activity that can be done to improve numeracy skills is to do matchstick puzzles with numbers with certain restrictions, such as only moving or removing 1 or 2 matchsticks to make up the equation or the sum. In the example below, you can add the restriction that you have to remove and move 1 matchstick to make the sum correct.



Another activity to practise addition or multiplication could be to play the find the sum that makes up the numbers. There are two ways to approach this, either saying the sum to the kids and giving them the option to choose how many different numbers using addition can make up the sum or giving characteristics of the number based on yes or no questions.

Such questions could be:

- Is the number higher than 20?
- Is the number odd (or even)?
- Can the number be divided by 2 (or 3)? (This question can be used for older kids).

After they find the number, you can backtrack and ask for possible number combinations.

This is also one of the convenient topics to introduce the game of *Which One Does Not Belong*. This game asks for four pictures or objects with a characteristic that three-by-three share a common attribute which allows for excluding the fourth item as the one which does not belong, but there is

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no one correct answer, as under the different conditions, every item could be excluded – you just need to find out the correct reason for it.

3+3	2+4
5+1	3+5

The Snake II

The game Snake II is played with 2 learners, where they roll the dice and have to move their tokens according to the value of the dice. Another way of approaching the activity for 6–7-year-old children is to use two dice in two different colours (e.g., red and blue), where red goes forward, and blue goes backwards. In this way, they get to practise subtraction.

Pat-a-cake counting introduces some rhythm into number patterns. Facing a child (or kids in pairs facing each other) you individually clap your hands together and then slap both of each other's hands. When you are good enough in this, add in a counting pattern that both of you say in unison while slapping each other's hands. For instance, you go clap, three, clap, six, clap nine, clap, twelve… Change turns in starting first and controlling the speed of clapping while the other one should keep up with the leader.

Or you could play the game of Twenty. It's a counting game in which two persons take turns in counting up from one to twenty with the aim of making your opponent say "Twenty". You are allowed to count one, two or three numbers. There is a strategy to win, but it is not obvious, although you could try with some hints to help children discover it. The target could be changed from twenty to bigger numbers, you could count in twos or threes, or even add a third player. Additionally, with younger kids you could use twenty coins, or tokens instead of counting the numbers aloud.

Counting Faces

In Counting Faces, learners throw the dice and have to find the shape that has the same number of faces as the dice. A way of extending this activity for younger ages to engage both in counting and geometry is to ask students to form a triangle by extending their arms. Once you form the triangle, you take a picture and ask the students to count how many arms are needed. To take it a step further, by looking at a tetrahedron, ask the students to count how many more would be needed to make the triangle into a tetrahedron.

For older ages, the activity can be extended by asking them to count the edges and vertices and observe what conditions are necessary for a shape to have vertices or edges. You can also ask them to recreate the platonic solids with magnetic sticks to be able to visualise the geometric shapes.

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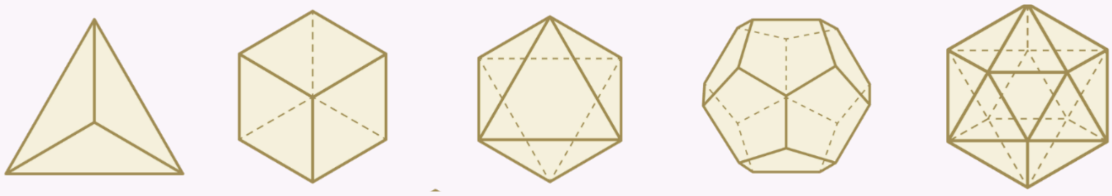
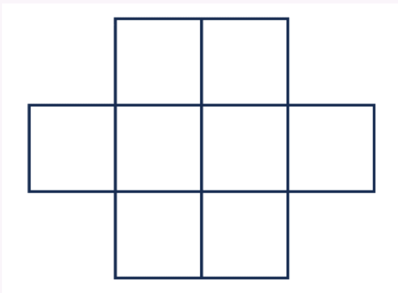


Happy Neighbours

This exhibit could be used by alternating the rules for the existing board, or by changing the board.

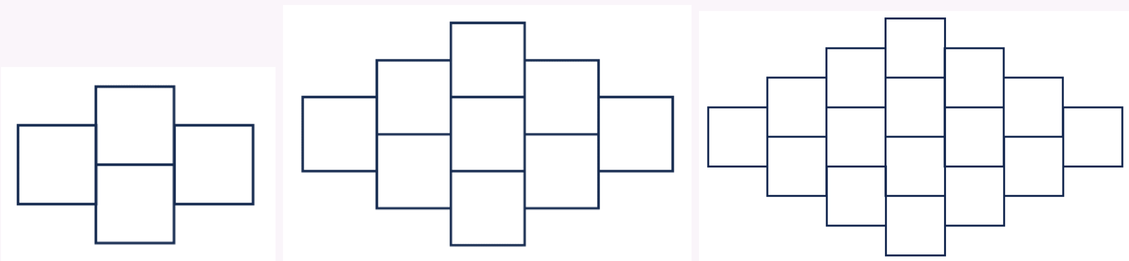
The first option would be to introduce tokens with numbers from 1 to 9 instead of colours but with a similar rule: consecutive numbers should not be neighbours. How many different ways are there to place the tokens by following this rule?

The boards could be changed in a way that the tasks gradually become harder to solve. The first option would be the board with only eight squares placed in the following way:



The rule would be to place tokens with numbers from 1 to 8 in such a way that consecutive numbers don’t share either side or vertices. For the younger children, instead of numbers, the tokens would be in three different colours, with three different options for the rules: same colours don’t share either side, not vertices, or 4 colour theorem – they cannot share side, but they can share vertex; how many colours are necessary in this case? If the rule is stronger, the same colours cannot share either side or vertices, we would need four colours, but with the rule that they can share a vertex, then three colours are enough.

The third version is to find out how to expand this exhibit with more squares while using three colours and the simpler rule. If we start with the simpler grid, with only 4 squares, can we populate them with the tokens in three colours, yes, or no? If yes, why is that so? Can we go on adding more squares? If we count the number of squares, the minimum is 4, then we have a grid with 9 squares, then 16, and then 25, which is actually the square of the number of squares in the central column of the grid.



Is there any relationship with Pascal's triangle?

How can we prove that the only version which satisfies the rules is the one with the grid consisting of the squared number of squares from the central column? There is a geometrical proof to it but is

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not graspable for children under 8 years of age. Simple material can contain high mathematics in disguise!

Families

The Families exhibit asks learners to sort objects into three different groups based on their own rules. An example of this would be size, colour and shape. There are many activities that can be done based on this idea. One activity could be to sort clothes for washing by colour or by their washing temperature.

Another activity could be to identify common characteristics between students and compare them, such as height, clothes, and long or short hair, by using a physical Venn Diagram and cards with categories to map out the commonalities. Afterwards, you can ask them to count how many are in which category.

A way of making it into a game would be to ask them to find other people who share the same attributes, such as age, height, and birth month. This game is called human bingo, and there are many templates that can be found online to create your own human bingo. An example is shown below by myfreebingocard.com:

Source: https://myfreebingocards.com/human-bingo

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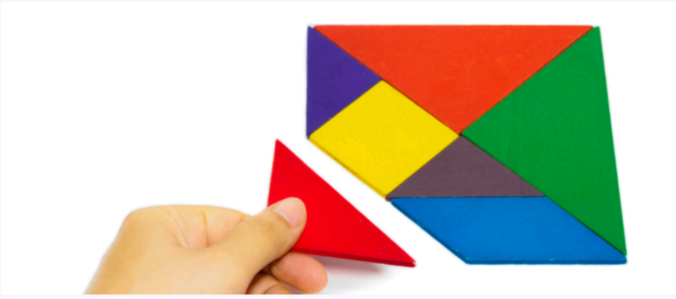
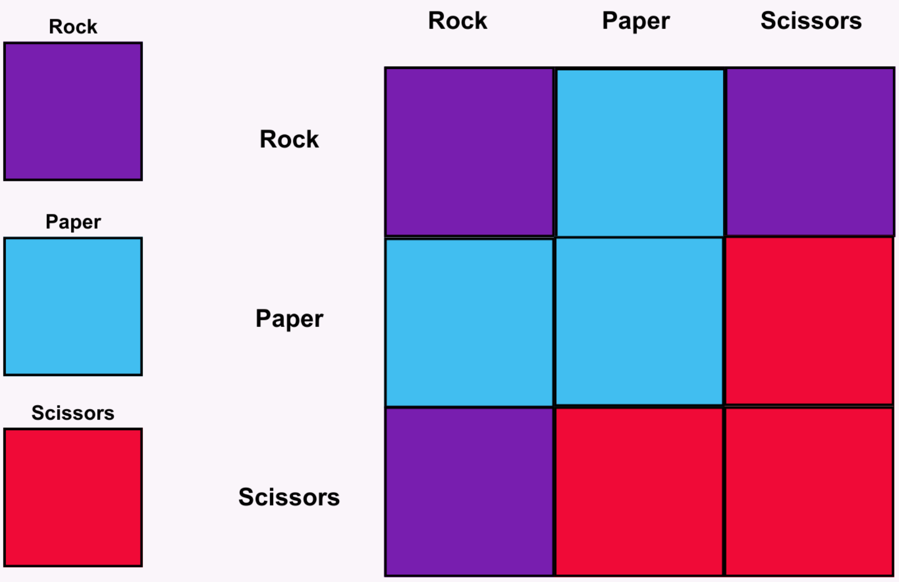
An activity based on this could be to sort students according to their birth month, which gives an introduction to probability. For instance, if there are more than 24 people, the chances are 50% that two people have birthdays on the same day.

A way of combining both the Families and Snake I (which deals with counting and a gentle introduction to probability) exhibits can be for teachers to make their own version of sorting shapes or objects under the condition that there is more than one family to which they belong. An example of this is logic blocks, which use the same shapes with different textures, sizes and colours of circles, triangles and rectangles. This is also something that can be done with the tangram puzzle.



The Snake I

As a way of extending the Snake I exhibit with the idea of heads or tails and working on activities which will make for introductory probability, you could ask students to play the Rock, Paper and Scissors game. You can assign a colour to each option and ask students to use the colour that wins according to each situation, and ask them to play the game and count how many times they won using one of the three.



How can a tennis match finish if you play in two sets or in three? This activity involves visually representing 2 and 3-won matches using drawings or stickers on a whiteboard, then inviting kids to participate in tallying the different ways a tennis match can end based on these conditions. Children take turns marking the board with various match outcomes, such as one player winning all matches, both players winning an equal number of matches, or different combinations of wins for each player. Through this tallying process, kids learn to count principles while exploring the diverse

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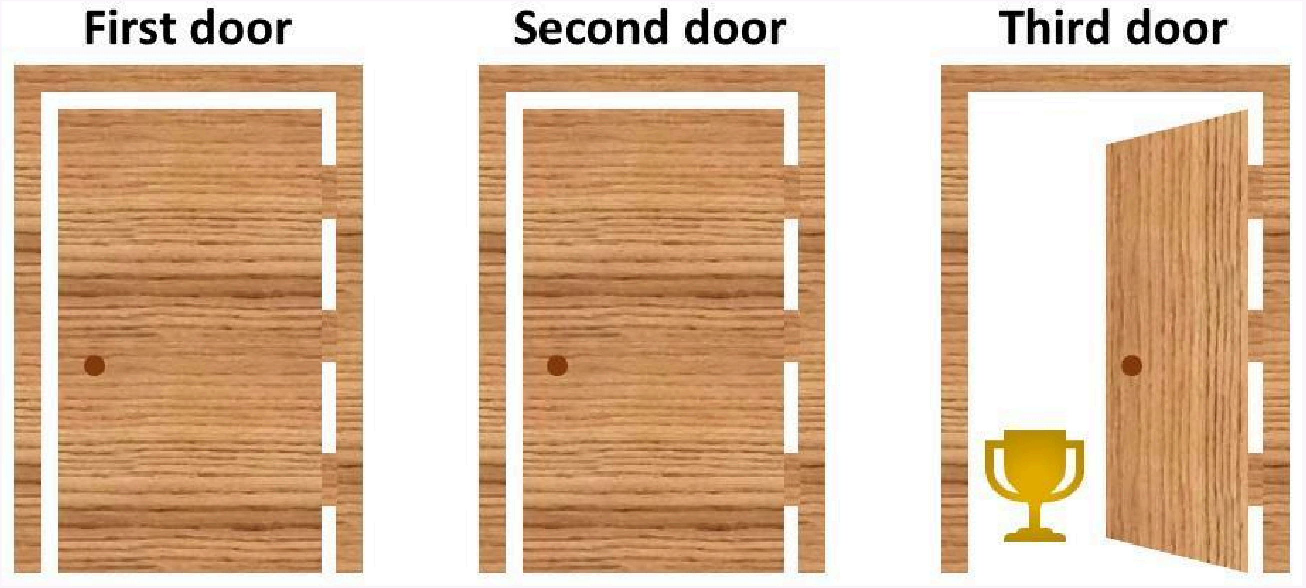
possibilities of how a tennis match could conclude, fostering engagement, creativity, and basic maths skills in counting and probabilities.

You can also play the Monty Hall problem as a game in the classroom where the presenter opens the door that they want, and the students have to choose if they will change their door or stay on the same one. The idea behind the game is that switching is always better than keeping the same selection. They can do a few rounds of the game in the classroom and count how many times they had to change their door. The explanation of the logic behind this problem can be done at a later stage.



Singing Birds

The Singing Birds app presents six birds ("lights") which can be on or off, and six mushrooms ("buttons") which can be pressed or not. Each mushroom switches the state of one or more birds, but we don't know in advance which birds are "connected" to each mushroom. Each game has different random connections. The birds are originally all off, and the goal is to switch them all on. Each bird in on state plays a note, all birds on make a nice chord.

The child will figure out these instructions, but finding a solution will come only after many trials and errors. After playing a certain number of rounds, the child could develop a few strategies, and the teacher could guide it to gain additional insights.

Before trying to solve a problem, we need to imagine the possible solution. The first key observation is that the order of pressing the buttons does not matter, and pressing a button twice yields the same result as not pressing it at all. This fact may not be evident at first glance, so one could imagine that the solution consists of a long pressing buttons sequence in a particular order. Once we

understand that order is unimportant, it is clear that we will solve this puzzle just by pressing certain buttons. Thanks to the nature of this solution, it is possible to create an enumeration strategy (for instance, to try all the possible combinations).

The second key observation is that, in general, some birds are affected by only a few buttons. If, by exploration, we find that one bird is only affected by one button, it will be conspicuous that the button is a part of the final solution. If we find that a bird is affected by two buttons, then one of them (but not both, and not none of them) must be turned on. This explanation separates these two buttons from the rest. We could test the first button, and if it does not yield the solution, then try the second one. These findings imply deductive reasoning that, in the end, reduces the combinations pool, thus narrowing the search for the final solution.

This trial-and-error technique improved by bits of deductive narrowing of the search space falls into what we call "observation and counting". Counting can be more convoluted than enumerating objects. First, it involves a description of the combination space, and second, it requires ruling out some (or all) invalid combinations that need not be tried.

For the interested reader, we include a mathematical description of the game. The problem can be formalized with linear algebra. We press the first button, we see which of the birds switch on, and we record this information as a column vector of six zeroes and ones (a 1 for each switched-on bird, a 0 otherwise). For instance, the vector (0,0,1,1,0,0) if only the third and fourth birds switch on). We repeat with each of the six buttons to get six column-vectors. We stack the six column-vectors to create a matrix A. Then, for any combination x of pressed buttons (for instance x = (1,1,0,0,0,1) as a column-vector if we press the buttons 1,2, and 6), the birds switched on are given by the matrix product $A \cdot x$. The matrices and vectors must be considered modulo 2, that is, substituting any even entry by 0 and any odd entry by 1. Solving the puzzle is equivalent to solving the linear system A · x = (1,1,1,1,1,1) T, which can be done by any standard method of solving linear systems of equations (Gaussian elimination, inverting A, etc). The program is made to guarantee that the matrix A is chosen to be invertible, so there is always a unique solution to the puzzle.

Other similar puzzles that can be proposed to children at similar ages include Lights-out, a physical electronics game popular in the 90s that can be played in online versions today. The mechanics are similar. In this case, the buttons are themselves also lights and there are 25 of them, arranged in a 5x5 grid. Each button toggles its own light and those of buttons adjacent to a side. This constant and predictable association between buttons and lights makes it easier to master with practice, recognizing patterns. A similar linear algebra study gives a mathematical solution.

Finally, symmetry and parity are among the most common and useful tools in mathematical magic activities (for instance, Baby Hummer or, for experts, the more general case of the Hummer Principle) or to find winning strategies in games (e.g. the game of NIM). The Marienbad version of Nim is a magnificent example of how symmetry and parity (applied to the binary system) allow you to win every game.

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Seaside Selfies

In terms of the location and position of objects in space, learners can be asked to match specific phrases with the position of objects in the picture. One example of this would be to use in front of and find which objects are in front of what. Expanding on this, the geoboard could be used to map out the location of the animals in relation to one another as a way of engaging in a Cartesian-like coordinate grid.

Moreover, different objects found in the classroom can be set up as a photography scene, where students will use the phone photography item that goes with the exhibit to take pictures from different perspectives. For older ages, this activity can also include light to learn and understand more about angles (where does the light come from, taking into consideration the shadows casted by different light sources).

Taking the same photo from different positions within the classroom and afterwards guessing by the photos who took them (this was taken from the left side of the classroom, or from the back of the classroom)

Taking the same photo with different ratios (1:1, 3:4, 9:16) and discussing how it affects the final photo in the sense of things included in it.

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Paths

Definition of Paths

The notion of a path is among the earliest concepts children encounter as they explore and navigate their surroundings. Visible paths crafted by kids may include stick traces drawn through sand, scribbles on paper, or wet marks left by a broken water bottle when moved around—the possibilities are endless. They might observe paths formed by fireworks, airplanes, shooting stars, animal tracks, and handwriting and even consider the idea of paths without visible traces.

In various settings, a path defines an area where movement is possible: one can traverse a street or a forest track or walk on a sidewalk but not through walls or buildings. Forest trails and countryside paths guide one's movement, cautioning against wandering into woods or fields, potentially leading to getting lost. There is a path between home and school that you can learn, and sometimes, you could return from school using a different route despite the same starting and endpoints. Learning a path requires directions—moving forward, turning right, navigating obstacles, or backtracking if blocked by an obstruction like a fence or closed door.

While an intuitive concept, defining a path involves two complementary perspectives: first, a dynamic definition interprets a path as the trajectory of a moving object (which may not leave a visible trace). Any moving entity—person, animal, car—traces a path during movement, absent when stationary. On the other hand, a static definition considers a path as a line (not necessarily straight) connecting two points in space, and no movement is needed. Both definitions exist in dictionaries, often accompanied by more symbolic meanings. Mathematically, both viewpoints are equivalent: the set of points an object traverses forms a curve, and given a curve, we can make a moving object follow it. However, introducing this duality to children can be engaging. Educators or parents can prompt discussions by posing a question: 'What is a path?' and presenting alternative perspectives. Combining these discussions with suggested activities can facilitate better understanding.

Exhibits from the SMEM Project related to Paths

Emy's Walk

The "dynamic view" is a good description of a path using a sequence of instructions: following step-by-step instructions while in motion outlines a path. This perspective finds a fitting illustration in the exhibit ***Emy's walk through the forest***. The exhibit features a checkerboard with numbered and coloured squares (see illustration). Each of the four colours directs movement towards a specific side, and the number indicates how many squares to go. Any child who can recognize numbers will easily follow these simple rules. Subsequent discussions might involve questions like "Which path did you follow?" prompting children to recreate their trajectory or provoke inquiries about the definition of a path itself. You could also ask: "Do you notice any path in the board"? This question encourages them to recognize that while an explicitly drawn path on the board (statically) does not exist, the rules define an implicit path.

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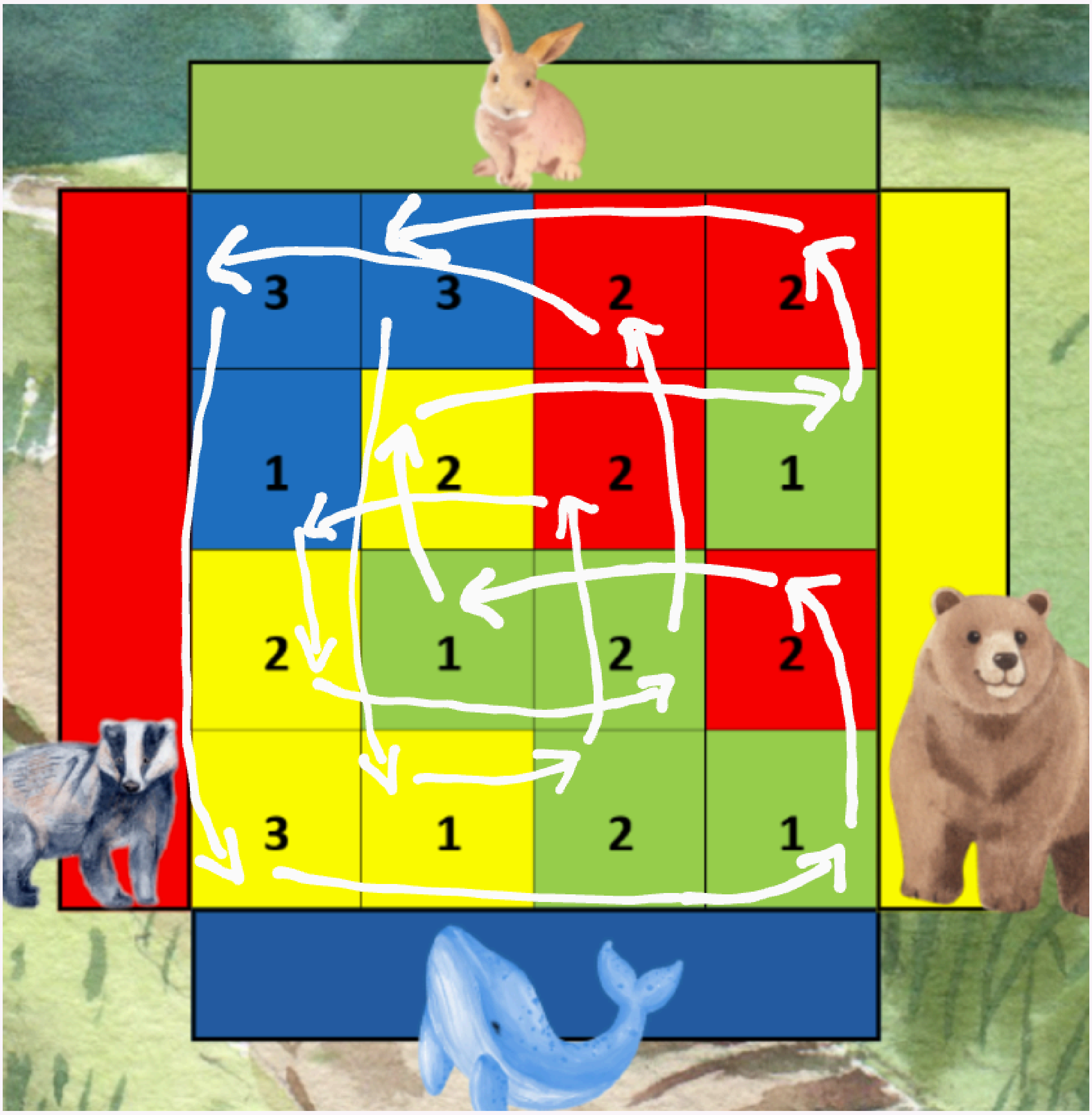

Engaging questions like "Is there a path adhering to the game rules between the yellow three and the blue one (or any other combination)?" or "Where should I begin my path to pass through..." can prompt exploration. Keeping the flow, questions like "Will my path return to its initial position?" introduce the concept of closed paths. Observations on self-intersections, directional terms (left and right, before and after, forward and backward), and discussions on path configurations can enrich the experience.

If you work with older children (8+), encourage them to construct their own checkerboard. They could do this from scratch by drawing the grid on paper or using pre-prepared squares already coloured and marked. You could also alter the instructions. For instance: "Your forest walk design should facilitate two, three, or four distinct looped paths." To help start the activity, you could partially pre-fill the structure, leaving a few squares blank for exploration and creation.

Heart in the sky

In the virtual exhibit Heart in the Sky, children control a flying kite to trace a predefined shape in the sky. While the target shape represents a 'static' path, the kite's trajectory embodies a 'dynamic'

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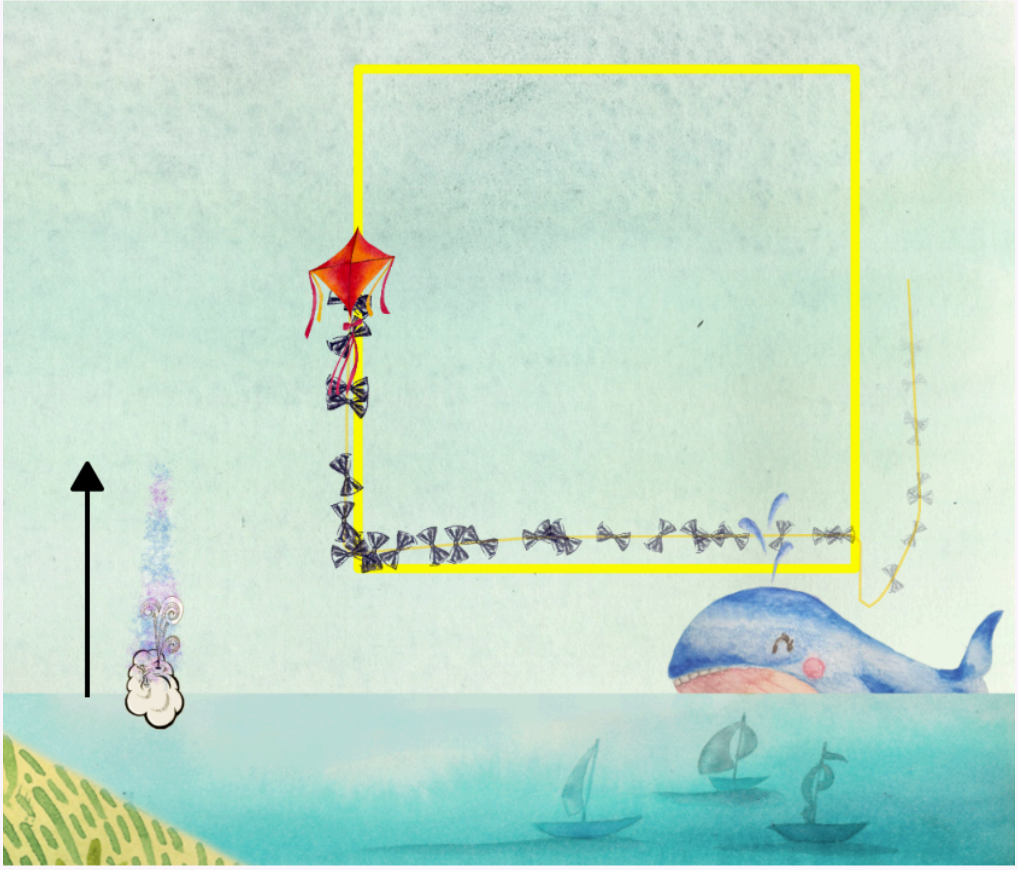
path. Interestingly, the kite's movement, despite being dynamic, leaves a trace on the screen, essentially turning the dynamic path into a static one. Here, the exhibit emphasizes direction independent of the path. Rather than directly controlling the kite, children guide it indirectly through a cloud-blowing wind, representing the kite's movement. This setup enhances eye-hand coordination and spatial orientation skills. As observed in the previous exhibit, directions play a pivotal role in determining the path. Furthermore, we introduce the concept of speed: the wind's direction indicates the direction of the kite's movement, and its length represents the speed. Together, direction and speed form the velocity vector of the moving kite, always tangent to the path.



Lily Pond

In the virtual exhibit 'Lily Pond,' paths emerge by connecting lilies with straight line segments. This exhibit extends the concept of paths to that of graphs: lilies represent vertices, and their connections form the edges of a graph. Each edge consistently links two vertices, ensuring no edges are disconnected or stray or an unconnected vertex exists. Vertices can serve as both the starting and ending points for multiple edges. Some edges might intersect. Overlapping edges are illustrated as thin lines, while non-overlapping ones appear as green strands. The original objective of the game involves untangling all the edges and eliminating overlaps. Yet, the displayed graphs offer an opportunity to explore paths further. By selecting a graph and labelling two lilies as A and B, one can investigate the various paths available to connect these designated points. How many different paths are there?

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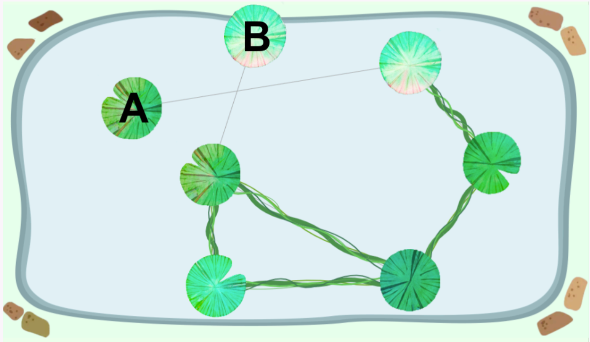
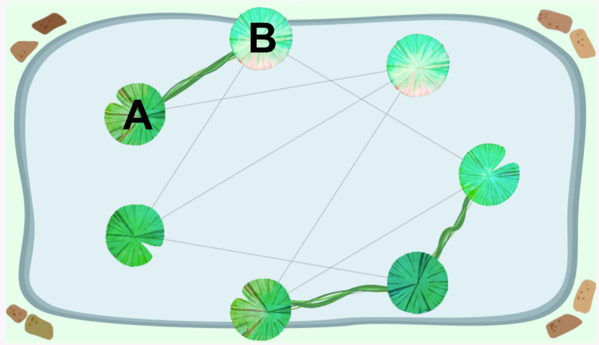
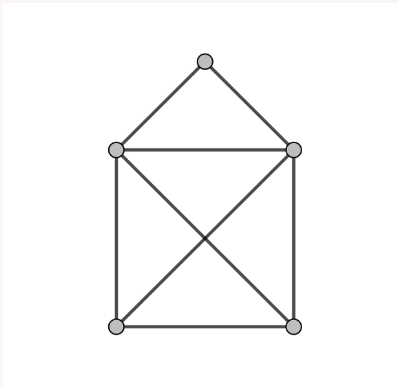




Depending on the selected graph, this task can vary from easy to more intricate. You can introduce additional conditions that the path must fulfil to be considered (such as non-intersecting lines, backward move allowed, etc.). Alternatively, challenge participants to find the shortest path connecting all lilies, akin to the travelling salesman problem.

Another engaging task involves connecting as many water lilies as possible while using each strand only once. Sometimes, it might not be feasible to connect all the lilies. The first picture demonstrates such a case. Then you could ask follow-up questions like: "Is it feasible to connect all lilies?", “Why or Why not?” "What conditions must be met to connect them all?" These tasks echo the famous problem of The Bridges of Königsberg. Children might also recognize a similar puzzle where they must draw a path on a graph resembling a house:



The Snake

As The Snake game unfolds, every player's token will trace a path across the board. Actually, a path resulting from the game automatically produces a graph (see the previous paragraph on the Lily Pond game). The illustration shows an example path for each version of the game.

You can ask kids how long the path of their token was. For the coin version of the game, it will be the number of turns it took to complete the game, while for the dice version, it will always be the same: the length of the snake (number of fields). In case the path on the board is not enough for the children to count to, they can draw a path of the appropriate length on a sheet of grid paper, utilising the squares as units for counting.

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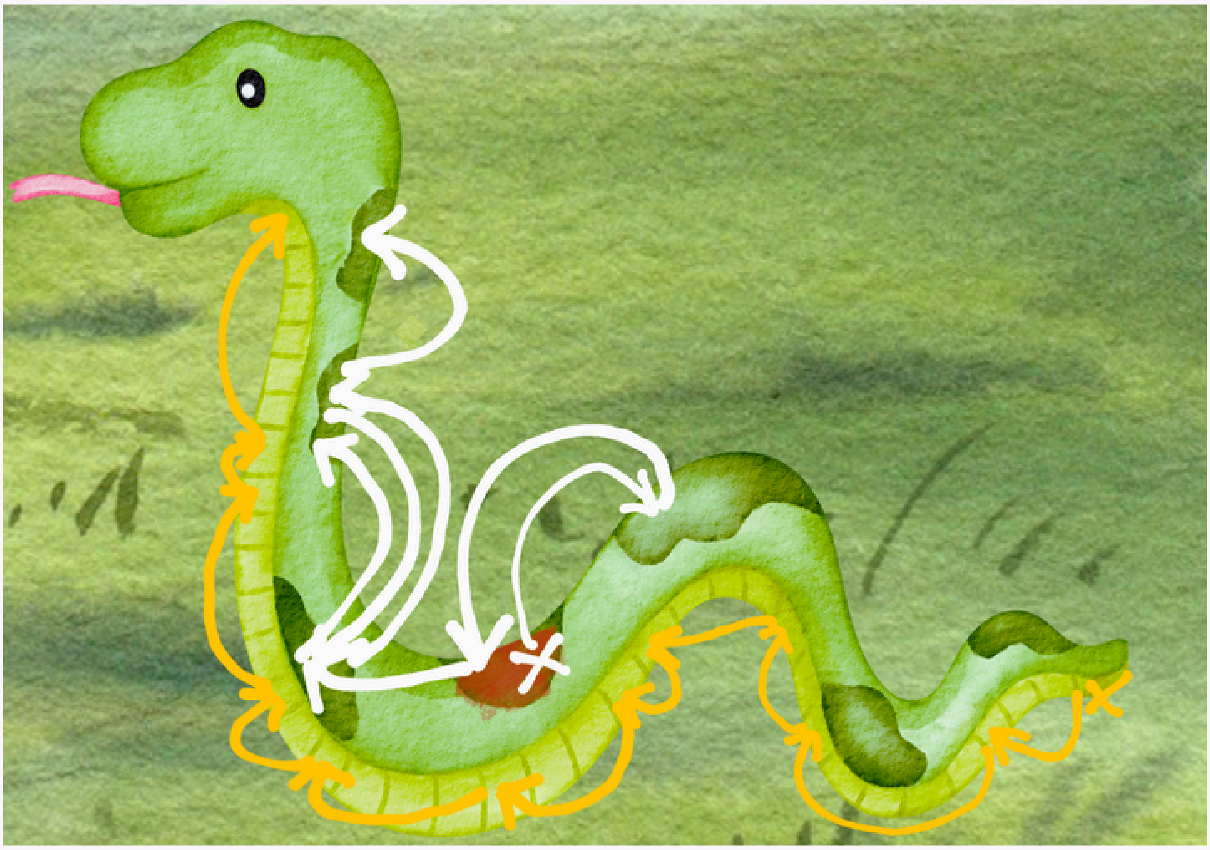
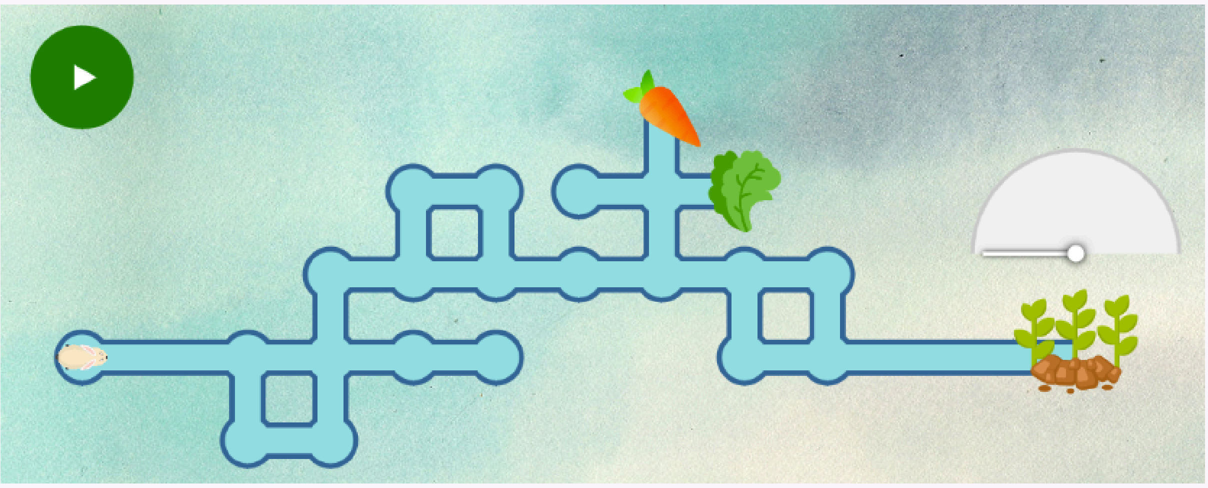


Rabbit’s labyrinth

The design of every stage within the virtual Rabbit’s Labyrinth exhibit forms a graph comprising vertices (round areas or intersections) and edges (connections between these rounded areas). Here is an example:



The rabbits navigate the graph, each following its unique route. Every path commences on the left side of the screen, concluding at either the Rabbit’s hole on the right side, a carrot, or a cabbage. These paths may differ in length and can contain multiple loops.

The objective is to modify the graph by strategically positioning rocks on vertices to ensure as many rabbits as possible reach the rabbit’s hole by the path’s end.

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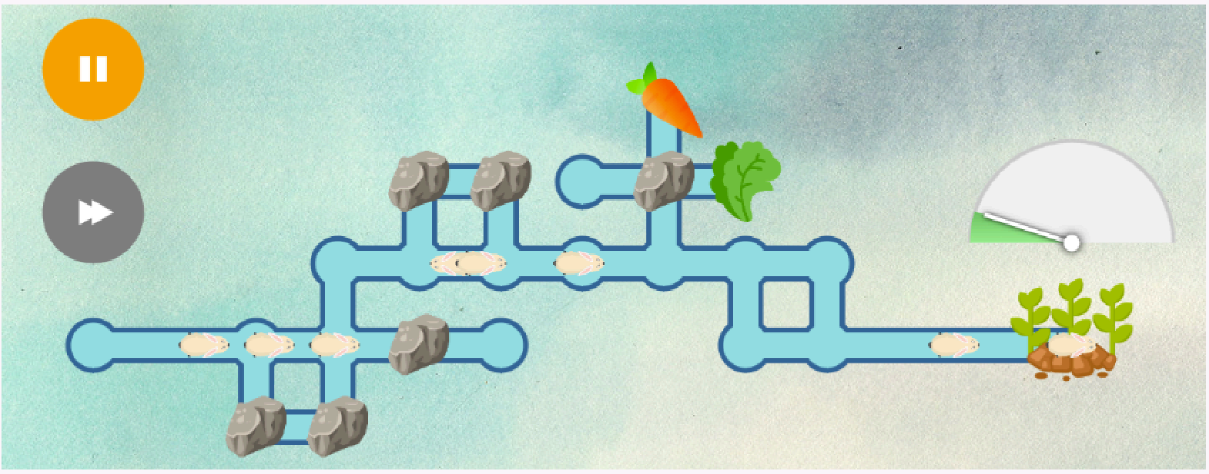
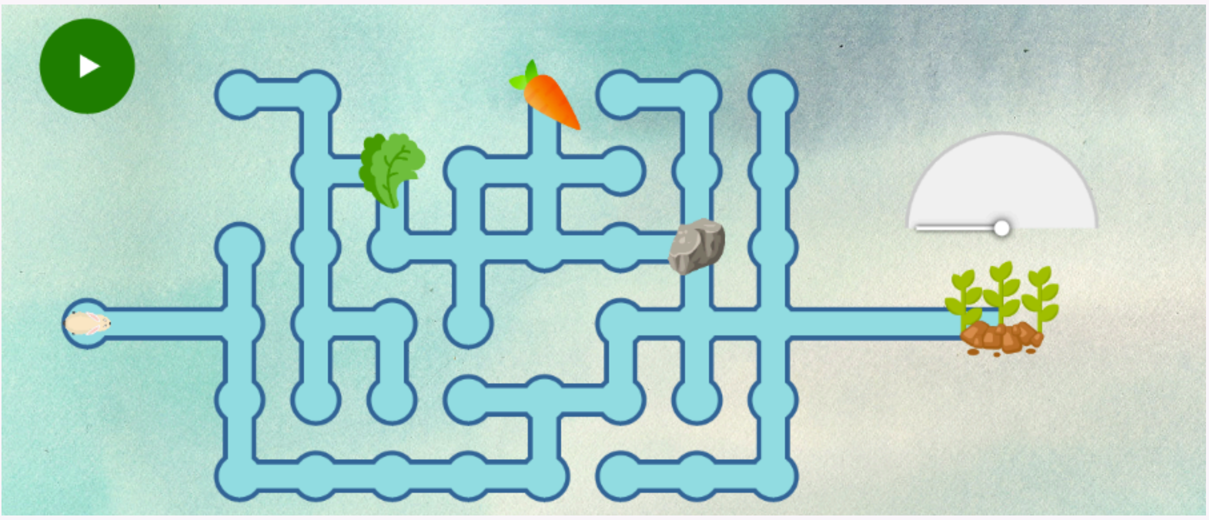


Rabbits can move along edges that lead to a rock but will be forced to reverse their direction upon reaching it. (Note: This behaviour differs from that of a mathematical graph as the remaining edge lacks a vertex at one end.) It's possible to block all three ending options, resulting in an endless game. Stones can also secure detours, aiding the rabbits in reaching their den swiftly (as seen in the provided screenshot: only one stone obstructs the carrot and cabbage; the other five aren't necessary for reaching the goal, yet they speed up the rabbits' arrival).

For children, initial questions about the graph (before they start placing rocks) might resemble those in the water lily game: "How many distinct paths can the rabbits follow?" or "What is the shortest path?" etc. Finding the shortest path will help in rock placement to achieve the original game's objective.

Then, you could ask about optimal rock placement to achieve the game's goal. Also, consider the minimum number of rocks required to block all distractions and their strategic placement in a graph you are playing. For instance, it's possible to obstruct two distractions with a single rock, even though it is feasible to block each distraction separately using individual stones.



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An Example of a SMEM-Based Workshop

In this section, we will explore a workshop based on activities from the SMEM project. The activities from the SMEM project serve as inspiration for creating dynamic learning experiences in the classroom and beyond.

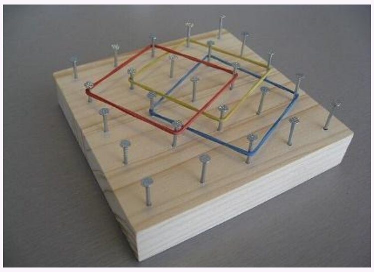
Age: 6-8
Workshop: Geometry and Spatial Awareness
Premises: Classroom / home environment
Time needed per activity: 20-25 mins
Activities: Seaside Selfies and Positional Understanding
Geometry Fun with Shapes
Discovering Geometric Patterns through Building

Materials needed: Cameras, geoboards, geometric manipulatives
Workshop includes: Match phrases with object positions, understand Cartesian concepts, explore shapes and their properties, build platonic solids, and experiment with angles and light through photography.
Maths topics covered: geometry, spatial concepts, angles, shapes

Seaside Selfies and Positional Understanding

This activity focuses on enhancing children's spatial understanding and positional awareness through a combination of visual activities and hands-on exploration.

Children are presented with images portraying seaside scenes containing various objects. They're prompted to match descriptive phrases with the corresponding positions of objects in these scenes. For instance, phrases like "beside the palm tree," "behind the boat," or "in front of the lighthouse" are paired with the respective objects in the images. This exercise helps reinforce their understanding of spatial prepositions and object placement.



To delve deeper into spatial concepts, children are introduced to Geoboards, a tactile tool resembling a Cartesian plane. They use rubber bands or pegs to create shapes or plot points on the geoboard. By doing so, they gain a practical understanding of basic Cartesian-like coordinate concepts such as X and Y axes, coordinates, and positioning objects in relation to grid points.

The session culminates in a fun and interactive photography activity. Children use cameras or smartphones to capture images from different perspectives—aerial, close-up, and panoramic views. They explore how changing the viewpoint alters the perception of objects' positions and sizes in the captured images. Discussions ensue, allowing children to express their observations about the effects of varied perspectives on object positioning and visual perception.

Throughout the activity, the facilitator guides discussions, encouraging children to articulate their understanding of positional terms, coordinate concepts, and how visual perspectives influence object placement. This open dialogue fosters critical thinking and allows children to connect real-world observations with geometric and spatial principles.

Concrete Examples of Prompts

1. Introduction to Coordinates

   Let's label the horizontal lines A, B, C, and the vertical lines 1, 2, 3 to create our grid. How does this help us locate points?
   Can you plot a point at A3? What coordinates would you use to plot a point on the grid?

2. Creating Shapes and Plotting Points

   Connect the points you've plotted. What shape have you created?
   Can you make a triangle using coordinates D2, E4, and F3? How would you plot these points?

3. Understanding Movements and Translations

   Let's move the square two units to the right and three units up. What will be its new coordinates?
   Describe the movement of the shape using coordinates. How did changing the coordinates affect its position?

4. Analysing Coordinate Relationships

   What happens if you change the second coordinate while keeping the first constant?
   Can you explain how changing the first coordinate moves the shape horizontally or changing the second coordinate moves it vertically?

5. Exploring Shape Properties and Transformations

   What happens if we connect A1, A4, D4, and D1? Can you describe the shape?
   If we mirror this shape across the vertical line at coordinate B, what will it look like?

6. Real-World Applications

   How might understanding coordinates help us navigate in a city or locate objects on a map?
   Can you think of situations where knowing how to use coordinates might be useful?

These discussions and guided questions aim to scaffold children's understanding of Cartesian-like concepts, encouraging them to think critically, articulate their observations, and relate these concepts to real-world scenarios. The facilitator's guidance prompts children to explore and visualise geometric concepts effectively using pen and paper.

As the activity concludes, a reflective session encourages children to share their insights and takeaways. They discuss how their understanding of spatial positioning has evolved and how this newfound knowledge might apply in real-life scenarios, reinforcing their grasp of geometric and spatial concepts.

This expanded activity emphasises the use of visual imagery, tactile tools like geoboards, and photography to enhance children's spatial understanding, reinforce positional concepts, and engage in discussions that bridge visual perception with geometric and spatial principles.

Alternative Activity: Cartesian-Like Concepts with Pen and Paper

This modified activity focuses on introducing children to basic Cartesian-like coordinate concepts using simple tools like paper and pencils.

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Children are provided with sheets of paper and pencils. They start by drawing a grid on the paper—a series of intersecting horizontal and vertical lines forming squares. The facilitator guides them in labelling the horizontal lines with letters (A, B, C, etc.) and the vertical lines with numbers (1, 2, 3, etc.), resembling a simplified Cartesian plane.

Using this self-made grid, children practise plotting points by choosing coordinates (e.g., A3, B4) and marking them on the grid. They proceed to connect these points to create shapes such as squares, rectangles, triangles, or more complex designs. Encouraging them to experiment with different coordinates fosters their understanding of how coordinates determine positions and shapes on the grid.

To further grasp positional concepts, children engage in activities involving moving shapes on the grid. The facilitator might instruct them to slide a shape (e.g., a square) from one position to another by specifying coordinates for its new location. This exercise reinforces the idea of how changing coordinates results in shifts or translations of objects on a visual plane.

As children work on their grids, the facilitator initiates discussions to explore the relationships between coordinates, movements, and the resulting shapes. Questions like, "How do changes in coordinates affect the position of the shape?" or "Can you describe the movement from point A to point B using coordinates?" prompt critical thinking and reinforce their understanding of spatial concepts.

Towards the end, a reflective session encourages children to share their experiences and observations. They discuss how working with coordinates and shapes on paper helped them visualise positional relationships and understand basic Cartesian-like concepts. The facilitator encourages them to think about practical applications of these concepts in everyday scenarios.

Geometry Fun with Shapes

This activity focuses on fostering children's understanding and exploration of various geometric shapes, their properties, and their relationships.

Children are presented with a variety of geometric shapes—circles, squares, triangles, rectangles, pentagons, hexagons, and 3D shapes like cubes, spheres, and pyramids. The facilitator initiates the activity by encouraging hands-on exploration and discussion around these shapes.

Concrete Examples of Facilitator's Guidance and Discussions

1. Introduction to Geometric Shapes

   Let's explore these shapes together. What do you notice about the properties of a square versus a triangle?
   How many sides does a hexagon have? Can you count and name them?

2. Experimenting with Shapes and Properties

   Can you build a shape using triangles that also has four sides? How?
   What happens when you try to fit two triangles together? Can you make them form a different shape?

3. Discussion on Symmetry and Patterns

   Look at this pattern made of squares and triangles. Can you identify the repeating elements?
   Can you create a symmetrical shape using only circles and squares?

4. Exploration of 3D Shapes

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Let's explore these 3D shapes. What are the differences between a cube and a sphere?
How many faces does a pyramid have? Can you count and name them?

5. Analysing Shape Properties

   Which shapes do you think can roll? Can you explain why?
   What makes a shape a 'regular' shape? Can you find examples of regular shapes around us?

6. Relating Shapes to the Real World

   Can you find examples of geometric shapes in the classroom or at home? Let's discuss their properties.
   How do shapes play a role in the structures we see around us, like buildings or furniture?

7. Encouraging Creative Exploration

   Create a unique shape using combinations of other shapes. How can you combine shapes to make something new?
   Can you invent a new 3D shape? What properties would it have?

This expanded activity aims to engage children in hands-on exploration, critical thinking, and deeper understanding of geometric shapes and their properties within a dynamic and interactive learning environment.

Discovering Geometric Patterns through Building

This activity invites children to a hands-on exploration of geometric patterns through building structures using various manipulatives. The session commences with a brief discussion on patterns, symmetry, and the use of shapes in construction.

Each child receives a set of building materials—such as wooden blocks, magnetic tiles, or interlocking cubes. The facilitator arranges the materials in accessible stations, ensuring a diverse range of shapes—squares, rectangles, triangles, and hexagons—are available for exploration.

Children engage in constructing geometric patterns, by replicating and extending given patterns or creating their own. The facilitator prompts them to experiment with symmetrical designs, alternating shapes, and creating sequences that repeat or grow progressively and poses thought-provoking questions to deepen their understanding. The kids are encouraged to articulate the rules governing their patterns, discuss symmetry, explore the relationship between shapes, and identify sequences within their designs.

Prompts and Guiding Questions

1. Replicating Patterns

   Can you recreate this pattern using different shapes?
   How many times does the pattern repeat itself?
   Can you extend this pattern to make it cover a larger area? Can you make it longer, wider?

2. Creating Symmetrical Designs:

   Can you build a pattern that's symmetrical along a line?
   How can you mirror this shape to create symmetry?
   Can you make the left side of your pattern match the right side?

3. Experimenting with Sequences

   What comes next in your pattern sequence?

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Can you create a sequence that grows by adding one more shape each time?
How can you change the sequence to double the number of shapes each time?

4. Identifying Relationships Between Shapes

   How do you decide which shape comes after another in your pattern?
   Can you create a pattern where each shape is half the size of the previous one?
   What happens if you rotate or flip the shapes in your pattern?

5. Encouraging Variation and Complexity

   Can you modify your pattern to include more shapes?
   What happens if you combine different shapes in your pattern?
   How can you make your pattern more intricate?

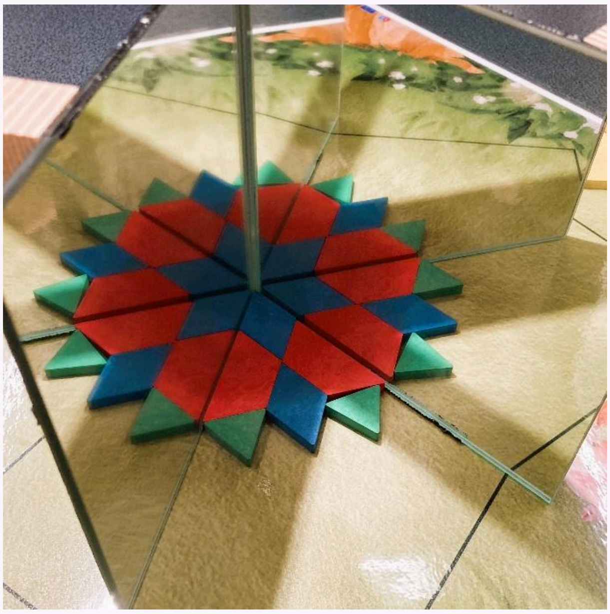
6. Discussing Pattern Properties

   What do you notice about the angles or sides of the shapes in your pattern?
   How many sides do your shapes have? Does it affect your pattern?
   Can you explain the symmetry or repetition you used in your design?

The activity fosters a collaborative environment where children share their patterns, allowing peers to identify underlying rules and extend the sequences collaboratively. The facilitator encourages experimentation, challenging children to create more complex patterns and explore variations.

Towards the conclusion, a reflective session unfolds. Children showcase their creations, explaining the patterns they've built, discussing the symmetry, repetition, and geometric properties observed. The facilitator guides discussions on the mathematical principles behind their patterns.

As the session draws to a close, children are encouraged to take home their manipulatives, allowing for continued exploration of geometric patterns. The facilitator shares suggestions for practising pattern creation at home, fostering an ongoing interest in geometric concepts.



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